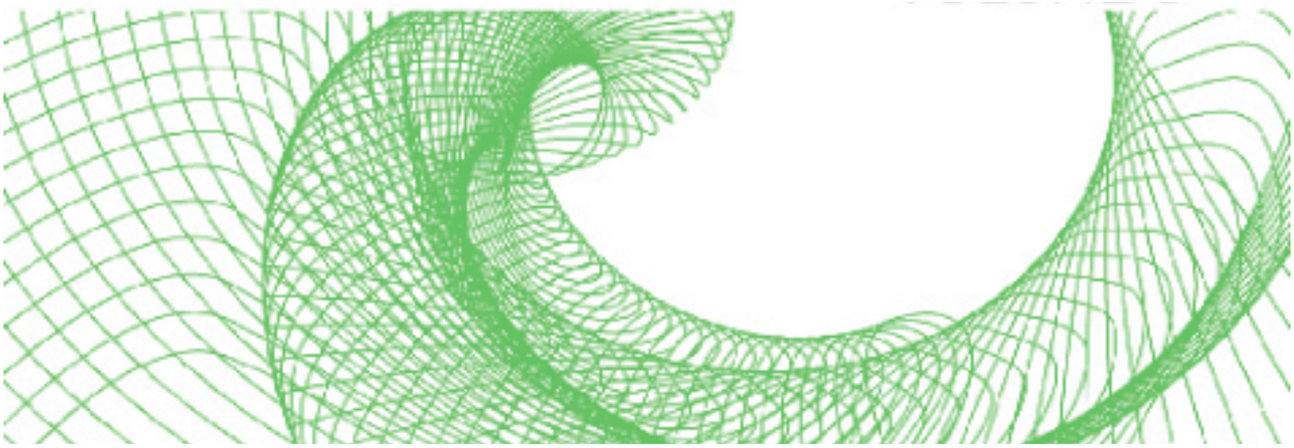
journal of the central california mathematics project



CALIFORNIA STATE UNIVERSITY, STANISLAUS

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Introduction

The articles in this journal emerge for the most part from mathematics classrooms. They are written by teachers, for teachers. They describe ideas and methods teachers feel are innovative, work for them, and want to share these nuggets of the teaching-learning process with other mathematics teachers. The writing style is informal, colloquial, and cursive. In this sense, the journal could be called a portfolio of teaching ideas for the mathematics classroom based on personal teaching experiences. There are some exceptions. We include some research articles, but even these are aimed at direct applications to help teachers in the classroom to teach mathematics more meaningfully.

The articles are blind reviewed by mathematicians and mathematics educators and accepted or not accepted based on their recommendations.

The reader will find the articles in this journal have some candid and not always complimentary statements on the existing state of mathematics education in the United States. And there are candid statements from now successful teachers whose own mathematics education could easily have turned them away from the subject they now dearly love to teach as a result of an insensitive teacher they might have had in their pre-college days. Some of these experiences described in this journal are based on the modus operandi of mathematics education indigenous in other countries, some from our own American classrooms. Whichever the case, the teachers who submit articles to this journal "have a story to tell;" a story about mathematics education.

We hope the reader will enjoy these articles and will also feel free to contact the editors with positive suggestions on how to make the journal better. We also encourage classroom teachers to submit a manuscript of their own for review and possible publication.

Thank you,

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Getting Students Excited About Learning Mathematics

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Introduction

It wasn't so long ago when mathematics students would diligently transcribe the words of wisdom of their professors without a hint of doubt. The popular phrase "just do it" has become an easy way out when things are not so easy to explain.

In many the Asian countries, mathematics students are not encouraged to ask questions. In fact, questions are considered impolite and a waste of time as they interrupt the class flow and decrease the amount of materials to be covered. Growing up in this culture, I was never curious enough to find out what powerful things mathematics could do in the real world. I never had an intrinsic motivation to learn mathematics, which I believe contributed to my mediocre performance on math tests. Little did I know that a single mathematics course later in life would alter my point of view so drastically and shape my teaching philosophy in such a profound way.

To me, the teaching and learning mathematics go hand-in-hand and always happen simultaneously. A teacher is not a walking library, which passively delivers information upon request; instead, a teacher plays the vital role of encouraging innovative thinking as well as stimulating curiosity. This can be done, for example, through the use of open discussions and hands-on projects.

What I like to do often times in the classrooms is to use various real-world problems to motivate the learning of certain concepts to get students excited about what these ideas could do before diving into the formality. And having a research area that is somewhat industrial-oriented, I have the privilege of bringing in a lot of interesting applications for this purpose.

For example: (Items 2 and 3 are not necessarily for high school.)

1) The idea of image sharpening can be used to motivate the learning of matrix multiplications.
2) The ideas behind Google's PageRank can be used to provide a practical use of eigenvalues and eigenvectors.
3) The ideas of image compression can be used to motivate the learning of matrix factoring such as singular value decomposition.
4) And the idea of handwritten digit classification can be used to motivate the study of tangent approximation.

In particular, I will use the rest of this article to provide a short illustration how one may use a real-world problem to motivate the learning of “tangent” approximation. I will do so by uncovering its uses at different grade levels as a way to increase students’ interest in learning the concept.

The take-home message for students as well as teachers is to challenge oneself in finding something that intrigues one in mathematics and see how far one can go with that idea. Hopefully, by doing so, students learn to discover more wonderful ideas in mathematics on their own.

An Illustrative Example

Some students first see the formal concept of tangent (or tangent line) in a high school geometry class where the tangent line (or simply the tangent) to a curve is the straight line that intersects the curve at one and only one point, (See Figure 1.)

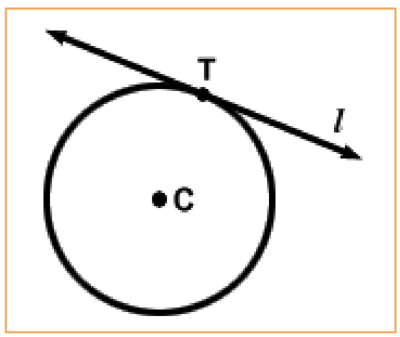
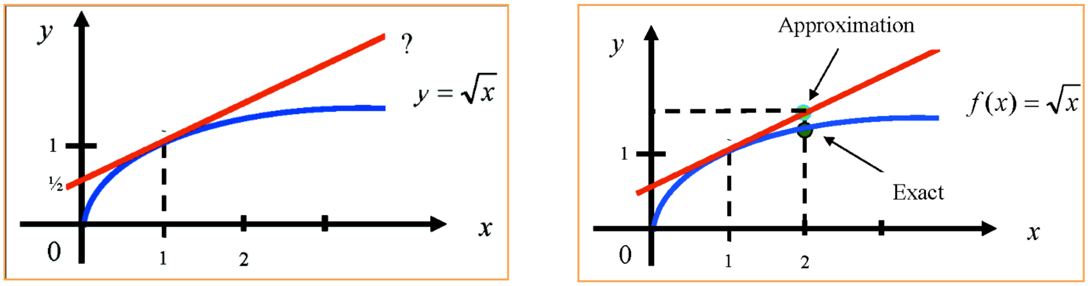


Figure 1. An illustration of tangent line to a curve at the point of tangency.

Other students first see the formal concept of tangent in a more algebraic setting where they are routinely being asked to find the equation of tangent lines without really seeing the purpose of doing so. As this point, it is important to convey the notion of tangent approximation through various non-polynomial functions, such as $y = \sqrt{x}$ as shown in Figure 2(a).

By obtaining the equation of the tangent line at $x = 1$, which is a first degree polynomial equation $y = f(x) = \frac{1}{2}x + \frac{1}{2}$, one can use it to approximate values such as $\sqrt{2}$ that are otherwise impractical to calculate by hand. With this, $\sqrt{2} \approx \frac{1}{2}(2) + \frac{1}{2} = 1.5$ is shown in Figure 2(b). Although the approximation is somewhat crude, considering that the actual value of $\sqrt{2}$ is about 1.414, it more or less provides a good starting point for further investigation.



(a) (b)

Figure 2

In order to decrease the error of such polynomial estimation, one may consider using higher-order polynomials such as quadratic or cubic polynomials, therefore arriving at the notion of Taylor approximation.

At the point of approximation, $x = a$, we have . . .

$$f(x) \approx \sum_{n=1}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^N(a)}{N!}(x-a)^N.$$

Notice that the truncation $f(x) = f(a) + f'(a)(x-a)$ resembles the equation of the tangent line discussed previously. Hence, for $f(x) = \sqrt{x}$ with $a = 1$, we can approximate $\sqrt{x}$ using the series

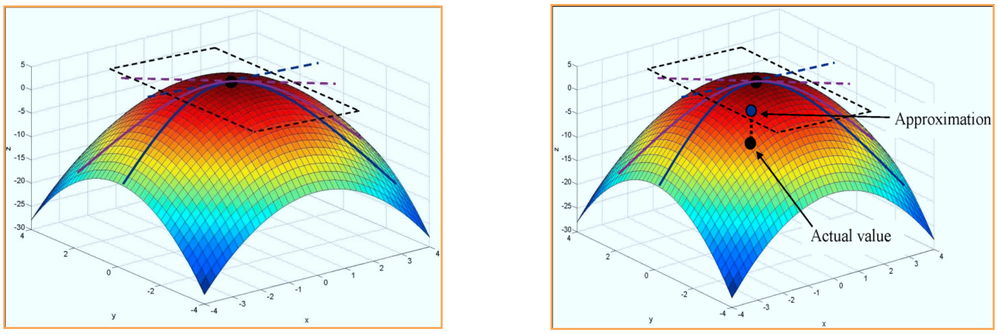
$$\sqrt{x} \approx 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4 + \frac{7}{2567}(x-1)^5 - + \cdots$$

For $x = 2$,

$$\sqrt{2} \approx 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} + \frac{7}{2567} - \frac{21}{1024} + \frac{33}{2048} - \frac{429}{32768} + - \cdots$$

One may choose to take as many or as few terms as needed depending on the desired accuracy. In fact, most of Texas Instruments (TI) calculators such as TI-89 Titanium and TI-Nspire use this series to calculate non-polynomial function values such as this one.

The idea of tangent approximations can be generalized in higher dimensions. A function in three variables whose graph, $z = f(x, y)$, is a surface in 3-dimensional space. The notion of tangent is no longer a line in this setting. Instead, there are infinitely many lines that are tangent to a point $(x, y) = (a, b)$ on the surface, as depicted in Figure 3(a).



(a) (b)

Figure 3. (a) An illustration of a tangent plane to a function at a given point. (b) Tangent plane can be used as a linear approximation to a function/surface for points near the point of tangency.

Together, these lines form a tangent plane $Ax + By + Cz = D$ to the surface at (a,b). The tangent plane gives a measure of linear approximation to the function and can be used to calculate function values for points relatively near (a, b) as shown in Figure 3(b).

As expected, the absolute error of approximation at (a,b), $\left| f(a,b) - \frac{D - Aa - Bb}{C} \right|$, increases as we move away from the point of tangency. As long as we stay relatively local, the tangent plane captures the variation afforded by the function and provides a convenient way to predict behaviors of the neighboring points.

Although our intuition fails us beyond three dimensions, the idea of linear approximation can be similarly extended to as many dimensions as we wish. In the high-dimensional analog, a surface is called a *manifold* if it looks like a plane locally along with a way to move from one patch to the next. A manifold is *differentiable* if we can do calculus on it. The "tangent" is formally called *tangent space* in higher dimensions.

The most common differentiable manifolds we often encounter are the spheres, $S^2: x^2 + y^2 = r^2$ – any small portion on the sphere looks like a plane and the latitudes and longitudes provide a recipe to relate points and give position.

Now, let us consider using the method of linear approximation to a real-world problem.

- Have you ever thought about how an ATM recognizes the dollar amount on a handwritten check?
- What about how the postal office sorts mails according to zip codes?

These two problems are considered under the umbrella of *handwritten digit recognition/classification*. You must wonder how these problems would have anything to do with manifold and tangent space. Before we proceed, a little background in the mathematics of digital images is in order.

A digital image of size $M \times N$ can be represented using a matrix of size $M \times N$ where each cell of the matrix contains a value between 2^0 and 2^k that gives different shades of gray and k is the number of *bits* a computer affords. For example, an 8-bit ($k = 8$) machine has 256 (0 –
255) shades of gray where a value of 0 means the pixel is completely dark while a value of 255 means the pixel is completely white.

Figure 4 shows how a simple black-and-white image is represented by a corresponding matrix of the respective size.

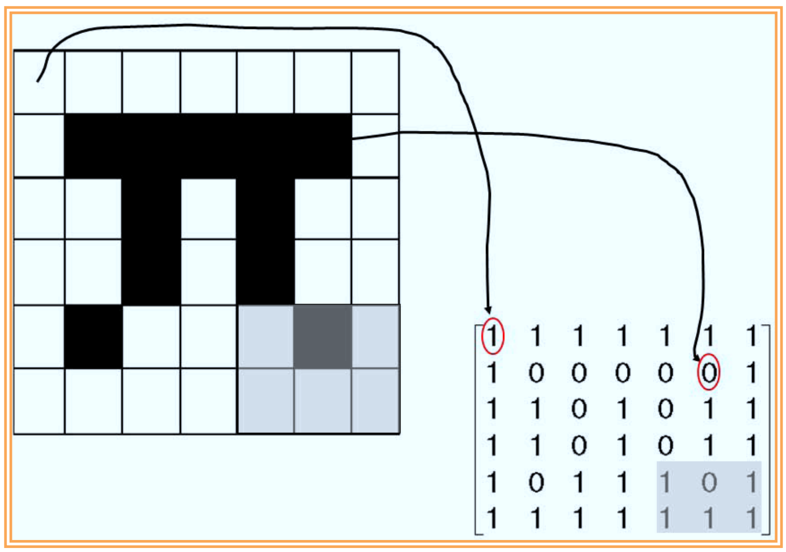


Figure 4. An illustration how an 8-bit image is represented by a matrix. In this example, black pixels are represented numerically by 0 while the white pixels are represented by 1.

Furthermore, we can concatenate an image matrix by columns so it can be viewed as a vector in MN-dimensional space. Precisely, the second column of the image matrix is moved underneath the first, the third column is then moved below the previous two, etc., as shown in Figure 5. This way, a 6×7 image corresponds to a vector in $\mathbb{R}^{42}$, a 42-dimensional space.

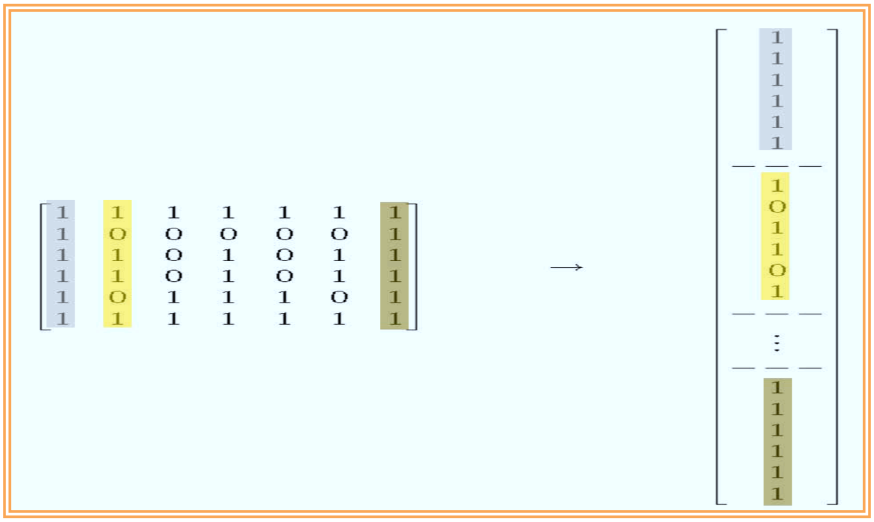


Figure 5. A matrix of size $\boldsymbol{M}\times\boldsymbol{N}$ can be viewed as a vector in $\boldsymbol{MN}$-dimensional space after column concatenation. In particular, $\boldsymbol{M} = \boldsymbol{7}$ and $\boldsymbol{N} = \boldsymbol{6}$ in this figure.

Thus, each monochrome digital handwritten image, in Figure 6, is a member of $\mathbb{R}^d$, where d represents the resolution of the images. Notice that $d = 32^2 = 1024$ in Figure 6.

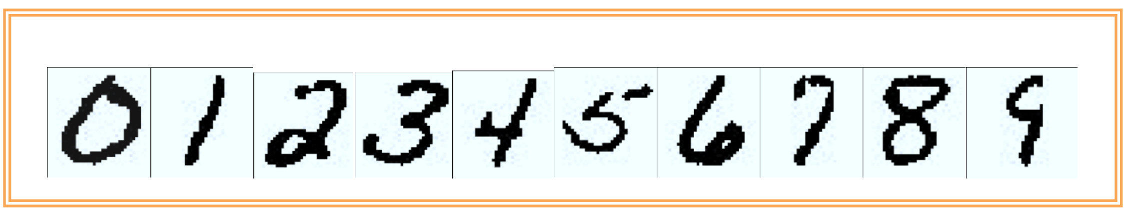


Figure 6. Example digits. Each digital image is of $\mathbf{32}\times\mathbf{32}$, therefore corresponding to a vector in $\mathbb{R}^{1024}$.

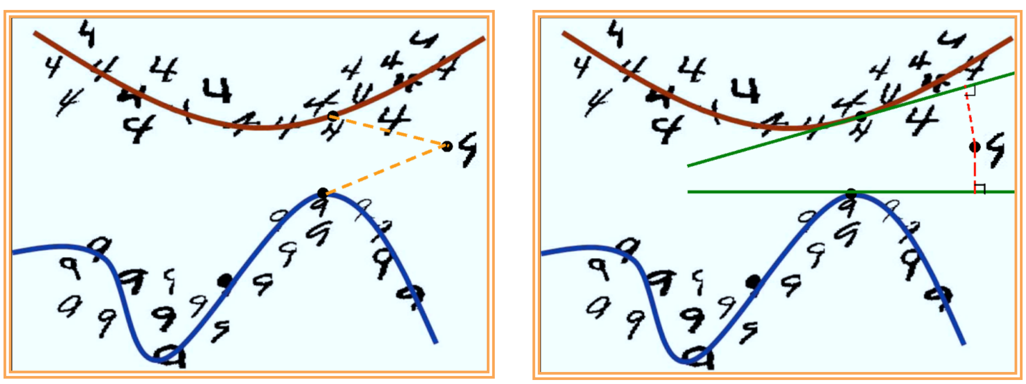
The problem of handwritten digit classification is the art of classifying an unseen digit using knowledge gained from a collection of labeled one, i.e., ones that we know the identity of. For simplicity, say we want to identify the unknown digit in Figure 7(a) given the known digits in Figure 7(b).

(a) (b)

Figure 7. (a) A probe digit. (b) Gallery digits.

The goal is to somehow learn the characteristics exhibited in the gallery patterns that are class-specific and can be used easily for assigning membership of the probe.

Geometrically, if we imagine all the 4's (from the training) live on some underlying manifold and all the 9's (from the training) live on another high-dimensional manifold, as depicted in Figure 8(a), then the straight-line distances between the probe and two random points in the gallery are shown as the dashed lines in Figure 8(a).



(a) (b)

Figure 8. The two curves passing through the digits are the digit manifolds from the training digits. (a) The Euclidean (straight-line) distance between the probe and the gallery digits are depicted as the dashed line. (b) The tangent distances between the selected gallery digits and the probe.

This straight-line distance is typically known as the Euclidean distance. Based on this distance measure, the probe would be labeled as the digit 4 since its distance to the selected digit 4 is less than the distance to the selected digit 9.

On the other hand, we can create a tangent space at each of the selected gallery patterns and calculate the tangent distance between the probe and these gallery points. As illustrated in Figure 8(b), the tangent distance between the probe and the selected digit 4 is the shortest path between the probe and all the points on the tangent space. Measuring distances this way, the probe would be classified as digit 9, which turns out to be the correct label. For a more detailed discussion, readers are referred to [1].

It is not hard to see that comparing a point to a space of points would work better than comparing it to just a single point since a space of points exhibit much more variation than a single point. This is why the tangent distance works better than the Euclidean distance for this problem.

Concluding Remarks

An important point to make here is that the given example is not to impress you with all the fancy things tangent space/distance can do; rather, the idea of taking a simple concept like the "tangent approximation" and seeing how far it can go is what we are really after.

In general, students tend to learn mathematics better when there is a concrete application to relate abstract concepts to. It is easier to engage students in the problem-solving processing when they are motivated by real-world problems that are particularly personal to them.

You might think that the ability to come up with real-world applications is somewhat limited to people who have backgrounds in industrial training. In fact, if we would just pause and try to think outside of the box for a while before we sit down and plan the day's lesson, there are numerous other ways to motivate the learning of mathematics that work for our individualized teaching styles. I challenge you to find something that works for you.

References

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Making Sense of Transformations of Graphs

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The topic of transformation of graphs supports student activities like looking for patterns, working with graphing technology, and making connections between different representations of functions. For example, students may look for patterns in the graphs of the following symbolically-represented functions:

$$f(x) = x^2$$
$$g(x) = x^2 + 1$$
$$h(x) = x^2 + 2$$

They may then be able to accurately describe the graphs as vertical translations of each other and be able to predict what the graph of $j(x) = x^2 - 1$ looks like. Indeed, the NCTM endorses activities such as these:

> *"Exploring functions of the form $f(x) = a(x - h)^2 + b(x - h) + c$ and seeing how their graphs change as the value of h is changed also provides a basis for understanding transformations and coordinate changes"* (NCTM, 2001, p. 300).

The preceding quote acknowledges that "exploring" and "seeing" only provide a basis for *understanding* transformations. That is, noticing and describing a pattern only gets us part of the way toward understanding why a pattern exists.

This article presents a way to make sense of the transformation of graphs of functions including:

- vertical and horizontal translations
- stretching and shrinking
- reflections.

This approach to transformations both relies on and deepens students' understandings of functions and their representations.

Functions and Their Representations

Understanding the transformation of functions and their graphs requires an understanding of functions. I will build upon the basic idea that a function is a relation between inputs and outputs where each input determines exactly one output. The variable representing the inputs is often described as the independent variable whereas the outputs correspond to the dependent variable (the output depends on the input). The symbols and representations that are used to describe functions reflect this idea. In the rest of this section I will be very explicit about the connection between the input-output relationship which characterizes functions and the various representations for functions; when teaching about transformation of graphs one of my primary goals is for students to strengthen their understanding of these same connections.

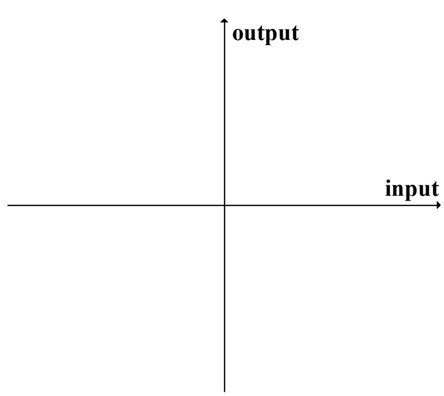
The notation $y = f(x)$ indicates that y is a function of x. That is, the input is called x and the output is called y or $f(x)$. We can refer to the function as f or as $f(x)$ or, if it exists, as the equation or rule which determines a function relationship between x and y. For example, we can think of the equation $y = x^2$ as determining y as a function of x. These ideas are reflected in the notation:

Output = *f*(Input)

$$\underbrace{y}_{output} = \underbrace{f\overbrace{(x)}^{input}}_{output}$$

We could represent all the input-output pairs that satisfy the function as ordered pairs (x, y). This notation can be thought of as (input, output).

The graph of the function f associates each of these ordered pairs with a point on the xy-plane. Each point is associated with a horizontal coordinate (the input) and a vertical coordinate (the output). The plane can be represented as:



Numerical representations are also connected to these most basic ideas about functions. The function $y = x^2$ could be represented with the following (partial) numerical representation:

Input	-2	-1	0	1	2	3
Output	4	1	0	1	4	9

This emphasis on input/output relationships and on the representations of functions will allow us to go beyond merely "exploring" and "seeing" how graphs are transformed. In the following sections I will present a way of understanding transformations.

Vertical Shifts

Vertical shifts are typically described according to the following rule:

> *Let f be a function and c be a positive integer. Then the graph of $y = f(x) + c$ is the graph of of $y = f(x)$ shifted upward by c units. The graph of $y = f(x) - c$ is the graph of $y = f(x)$ shifted downward by c units.*

In general, the set of rules for transformations is fairly easy to memorize for transformation of graphs. If forgotten, the rules can quickly be re-established with a few examples on a graphing calculator. Unfortunately, this could be a disincentive to grappling with and building understanding of the ideas which make these rules true.

Furthermore, the study of transformation of graphs is an opportunity for students to deepen their understanding of functions and their representations. In the context of vertical shifts, we may examine a function like $g(x)=x^2+1$ as a transformation of $f(x) = x^2$. In other words, $g(x) = f(x) + 1$.

The first question I would ask my students is: *Have we done something to the input or to the output of f?* The key realization is that we have added one to the output, $f(x)$, so we would expect something to happen in the vertical dimension of the graph of *f*.

The structure of $g(x)$ tells us that, for the same input, *g* assigns an output one greater than *f* assigns. If $(a, f(a))$ is any point on the graph of *f* then $(a, f(a) + 1)$ is a point on the graph of *g*. This means that the graph of *g* looks just like that of *f* shifted up by one unit. This can be seen in the numerical representation as well.

x	-2	-1	3	1	2	3
$f(x) = x^2$	4	1	0	1	4	9
$g(x)=f(x)+1$	4+1	1+1	0+1	1+1	4+1	9+1

These various representations all reveal the following fact about the function $g(x) = f(x) \pm c$: with the same input, the output of g will be c units larger or smaller than the output of f.

Horizontal Shifts

Horizontal shifts can also be understood in terms of the input-output relationship which characterizes functions. Investigative activities with graphing calculators can reveal the rule:

> *Let f be a function and c be a positive integer. Then the graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted left by c units. The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted right by c units.*

Thus, a function like $g(x) = (x + 1)^2$ is a transformation of $f(x) = x^2$. That is, $g(x) = f(x+1)$; f is the function which squares its input and the input-output relationship given by $g(x) = (x + 1)^2$ is the same as using that determined by using $x + 1$ as the input for f.

Again, we start with the question: Have we done something to the input or to the output of f? We have added 1 to the input, x, so we would expect something to happen in the horizontal dimension of the graph of $f(x)$.

The equation $g(x) = f(x + 1)$ tells us, for example, that $g(0)$ is the same as $f(1)$, $g(1)$ is the same as $f(2)$, and so on. If we use an input for $g(x)$ that is one less than the input for $f(x)$ then we get the same output. So for every point $(a, f(a))$ on the graph of f, the point $(a\text{-}1, f(a))$ is on the graph of g; note that $g(a\text{-}1) = f(a)$. This means that the graph of g looks like the graph of f shifted to the left by one unit.

x	-3	-2	-1	3	1	2	3
$f(x) = x^2$	9	4	1	0	1	4	9
$g(x) = f(x+1)$	4	1	0	1	4	9	

In general, if $g(x) = f(x + c)$ then we can use $a\text{-}c$ as an input for g and the output will be $f(a)$. An input c units smaller will yield the same output.

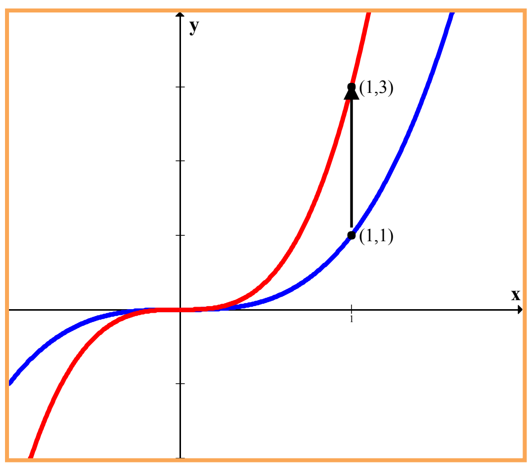
Stretches and Shrinks

We now turn our attention to horizontal and vertical stretches and shrinks. Once again, when presented with the symbolic representation of a transformed function we will first ask the question:

> *Have we done something to the input or to the output of the original function?*

In the function $g(x) = c \cdot f(x)$ the output of f has been multiplied by c. We should thus expect something to happen to the graph in the *vertical* dimension. That is, for the same output, $g(x)$ will give us the output of $f(x)$ times c. So if $(a, f(a))$ is on the graph of f then $(a, cf(a))$ is on the graph of g for any real number c and any a in the domain of f.

For example, if $f(x) = x^3$ and $g(x) = 3f(x) = 3x^3$, then the graph of $g(x)$ will be the graph of $f(x)$ *stretched* vertically by a factor of three.



When $0 < c < 1$, the graph of $f(x)$ is *shrunk* vertically. Consider, if $f(x) = x^3$ and $h(x) = ½\cdot f(x) = ½\cdot x^3$. Given the same input, the output of $h(x)$ will be one half of the output of $f(x)$.

X	-2	-1	0	1	2
f(x) = x³	-8	-1	0	1	8
g(x) = 3f(x)	-24	-3	0	3	24
h(x)= ½f(x)	-4	-½	0	½	4

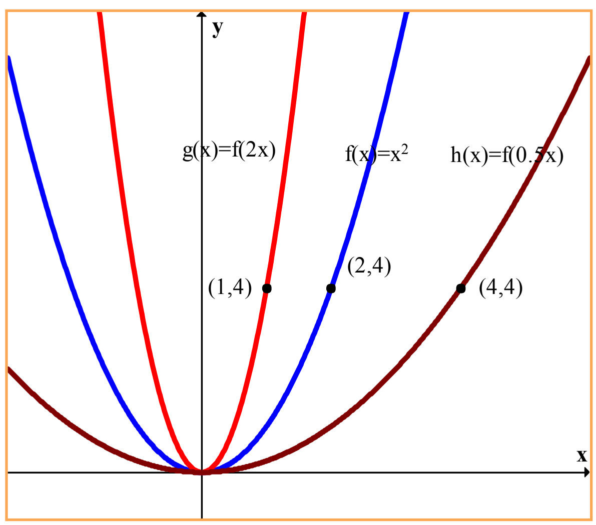
What about $g(x) = f(cx)$? Here we have done something to the input so we would expect something to happen to the graph horizontally. The structure of g tells us that f and g will have the same output if the input for g is $1/c$ times as large as the input for f.

For example, if $f(x) = x^2$ and $g(x) = f(2x) = (2x)^2$ then $g(½) = f(1)$, $g(1) = f(2)$, $g(3) = f(6)$, and so on. In general, if the point $(a, f(a))$ is on the graph of f then the point $(a/c, f(a))$ is on the graph of g for any nonzero number c and any number a in the domain of f. Thus:

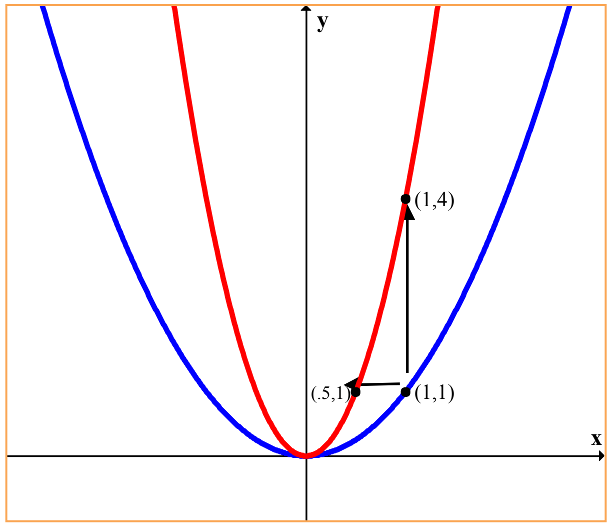
- *if* $c > 1$ *then a smaller input for g yields the same output as f; that is, the graph of g is the graph of f shrunk horizontally.*

- *if* $0 < c < 1$ *then a larger input for g yields the same output as f; the graph of g is the graph of f stretched horizontally.*

The graph below shows $f(x) = x^2$, $g(x) = f(2x) = (2x)^2$, and $g(x) = f(½x) = (½ x)^2$. The point (2, 4) on the graph of f corresponds to the point (1, 4) on the graph of g and (4, 4) on the graph of h.



It's worth noting that, for some functions, stretching vertically is the same thing as shrinking horizontally. Likewise, stretching horizontally is often the same thing as shrinking vertically. For example, if $f(x) = x^2$ then $g(x) = f(2x)$ shrinks $f(x)$ horizontally and $h(x) = 4f(x)$ stretches $f(x)$ vertically according to the observations made above. Thus in this example we have $g(x) = f(2x) = (2x)^2 = 4x^2 = 4f(x) = h(x)$; i.e., $g(x) = h(x)$. This is perhaps easier to see in the graph below; it shows that we can think of the point (1,1) on f as being vertically stretched to (1,4) or horizontally shrunk to (½,1).



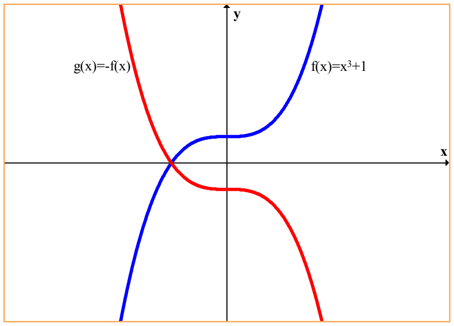
Reflections of Graphs

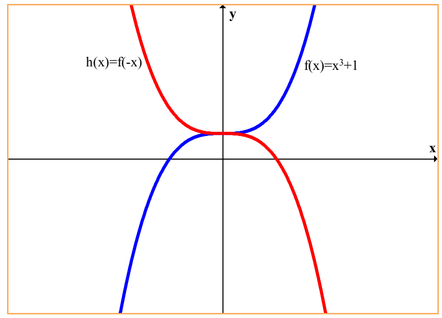
In the preceding section we did not see any examples where *c* is negative. It is important to acknowledge that the reasoning we used to understand stretches and shrinks does not change whether *c* is positive or negative. However it is also worth describing this scenario in terms of reflections across the *x*- and *y*-axes. We'll consider two examples:

$$g(x) = -f(x) \text{ and } h(x) = f(-x); \text{ i.e., } c = -1$$

We can describe the function $g(x)$ as follows: if you use the same input for *f* and *g* the outputs will always be opposite in sign. Since the outputs have been affected we should expect to see a vertical transformation. Indeed, since the same input yields opposite outputs for *f* and *g*, the graph of *g* looks like the graph of *f* reflected over the *x*-axis. If the point $(a, f(a))$ is on the graph of *f* then $(a, -f(a))$ is on the graph of *g*.

With $h(x)$, the input of *f* has been altered so we would expect a horizontal transformation. If you use opposite inputs for *h* and *f* then you get the same output. That is, if the point $(a, f(a))$ the graph of *f* then $(-a, f(a))$ is on the graph of *h*. The effect is that the graph of *h* is the graph of *f* reflected across the *y*-axis.

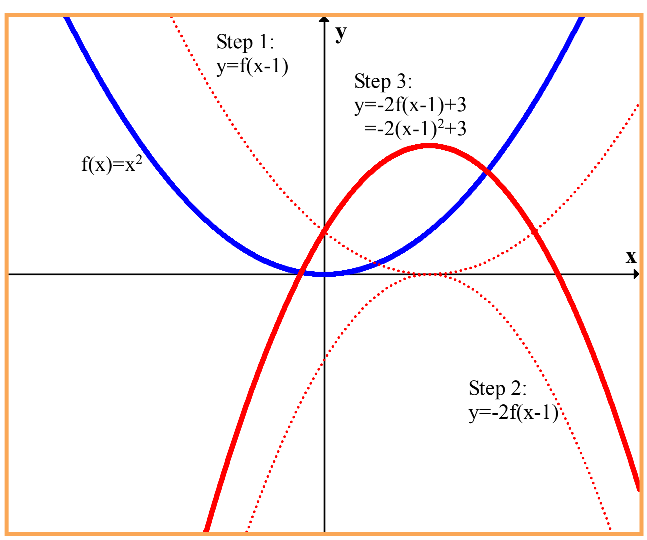




Combining Transformations

The reasoning we have used can be applied to more complicated transformations. For example, if $g(x) = 3f(x+2) - 1$ then, following the order of operations, we may conclude that the graph of g looks like the graph of f shifted left 2 then stretched vertically by a factor of 3 then shifted down 1. Moreover, if the point (a, b) is on the graph of f (i.e., $f(a) = b$) then the point $(a - 2, 3b - 1)$ is on the graph of g.

This thought process can be applied to quadratic functions; i.e., functions which can be written in the form $f(x) = ax^2 + bx + c$, $a \neq 0$. These functions can also be written in "vertex form". For example, the function represented as $g(x) = -2x^2 + 4x + 1$ can also be represented as $g(x) = -2(x - 1)^2 + 3$ by completing the square. The structure of $g(x) = -2(x - 1)^2 + 3$ indicates that $g(x)$ can be thought of as a transformation of the function $f(x) = x^2$. That is, the graph of $g(x)$ looks like the graph of $f(x)$ shifted over 1 unit to the right, flipped over the x-axis and stretched vertically by a factor of 2, and then shifted up 3. This means that the vertex must be at (1,3).



Since we can write any quadratic function in vertex form we can always think of a quadratic function as a transformation of $f(x) = x^2$. This is a convenient way to understand some basic facts about the graphs of quadratic functions. For example, we may conclude that all quadratic functions have parabolas for graphs. We may also conclude that the sign of the leading coefficient determines whether the graph opens upward or downward. Furthermore, we can see that transforming a parabola can result in 0, 1 or 2 zeros.

Concluding Remarks

I have taught transformation of graphs to students at various levels. I have found that, with a little bit of structure, students are generally successful at exploring transformations and making correct conjectures. The discussion above presents a way to help students make sense of and justify those conjectures. Moreover, it presents an opportunity for students to strengthen their understandings of functions and the multiple representations of functions. For example, the question of whether a transformation is vertical or horizontal depends on whether we have altered the output or input of the original function. It is important for students to “explore” and to “see” the patterns which emerge in the study of transformation of graphs; it is even better to encourage their mathematical curiosity and to help them make sense of why these patterns exist.

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WeBWorK - Part II

Perceptions and Success in College Algebra

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Introduction

This is the second of a three-part article meant to familiarize practicing teachers with WeBWorK, an open-source web-based software designed to support students' out-of-class attempts in mathematics learning. Used in many U.S. schools and colleges, the software (a) presents mathematical exercises, problems, or tasks, (b) students work out the problems, preferably with paper and pencil to one side of a computer, and (c) enter their solutions into the computer, that is, into the window provided by the program. [1] WeBWorK then gives immediate feedback ("correct" or "incorrect") to the student, but the interface does not correct a student's errors or give hints or point to conceptual flaws in case of an incorrect answer. [2]

If students need help, they are encouraged to seek out a fellow student, a tutor, or the instructor. They can do this in person or by email. For more on how this particular open-source free program is used by teachers and students, see Part One of this report in this journal; Segalla and Hauk (2010).

Research results

In this second installment of our reporting about WeBWorK, we share results from a study of college algebra classes where WeBWorK was used as a substitute for paper and pencil homework. Students in 12 of 19 classes had homework problems to be completed through the web-based software WeBWorK, while students in the other 7 of the 19 classes were assigned the same prob-

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[1] WeBWorK has the ability to accept (and evaluate) mathematical notation. For example, it will accept a function as an input if that is what the answer requires.

[2] The instructor can program general hints in each problem set; anticipatory hints that may guide the student to the correct solution. Example: "The quadratic equation you are solving is not factorable over the rational numbers."

lems – all were from the course textbook – to be completed using the traditional pencil-and-paper approach. First, the bottom line:

> *Student achievement in the web-based homework group was*
> ***at least as high*** *as the achievement in the paper-and-pencil group*.

That is, even a narrow use of WeBWorK as a substitute for handwritten homework can be at least as effective as traditionally graded paper and pencil homework for students learning the mathematics common to the high school second year of algebra.

Perhaps as important is the fact that WeBWorK challenged students and some teachers to break some research-based proven perceptions of how students and some teachers feel about mathematics, or better, what they believe that mathematics is about. All indications are that WeBWorK may productively challenge, and hopefully change, some of the detrimental beliefs about mathematics learning and teaching.

Our focus for this report was the first-year college algebra classes at a California state college we will call Big Public University (BPU; see Figure 1 for an overview of the student demographics compared to national averages).

Our questions

1. Student Achievement

Given that the same homework items were assigned in web-based homework (WBH) and paper-and pencil homework (PPH) sections, and controlling for preparedness by way of pre-test and national norm-referenced tests (SAT-Math and SAT-Verbal), how did student achievement in the two situations compare?

2. Student Perception

Among the students who did web-based homework, what are perceptions of the nature, purpose, and use of web-based homework, particularly of their efforts and degree of success using WeBWorK?

3. Instructional Style

What contributions to differences in students' perceptions and performance might be attributable to instructor style?

In this study:

- 644 students were enrolled in the 19 class sections.
- Of these, 532 (84%) completed the course while 112 (16%) dropped or withdrew.
- Of the 532 who finished the course, 435 (82%) passed it with a D or better:
  A (19%), B (28%), C (24%), or D (11%).
  That is, of the 644 who originally enrolled, 435 passed, 97 failed, and 112 withdrew from the course.
- There were no statistically significant differences in these percentages between the WBH and PPH sections.
- There were 408 students in the 12 WBH sections and 236 students in the 7 PPH sections.
- Fifteen instructors taught the 19 classes. Each of the three instructors who taught multiple sections of the course had at least one PPH and one WBH section.

Figure 1 below compares the diversity of students in this study (Big Public University) with that of the national average.

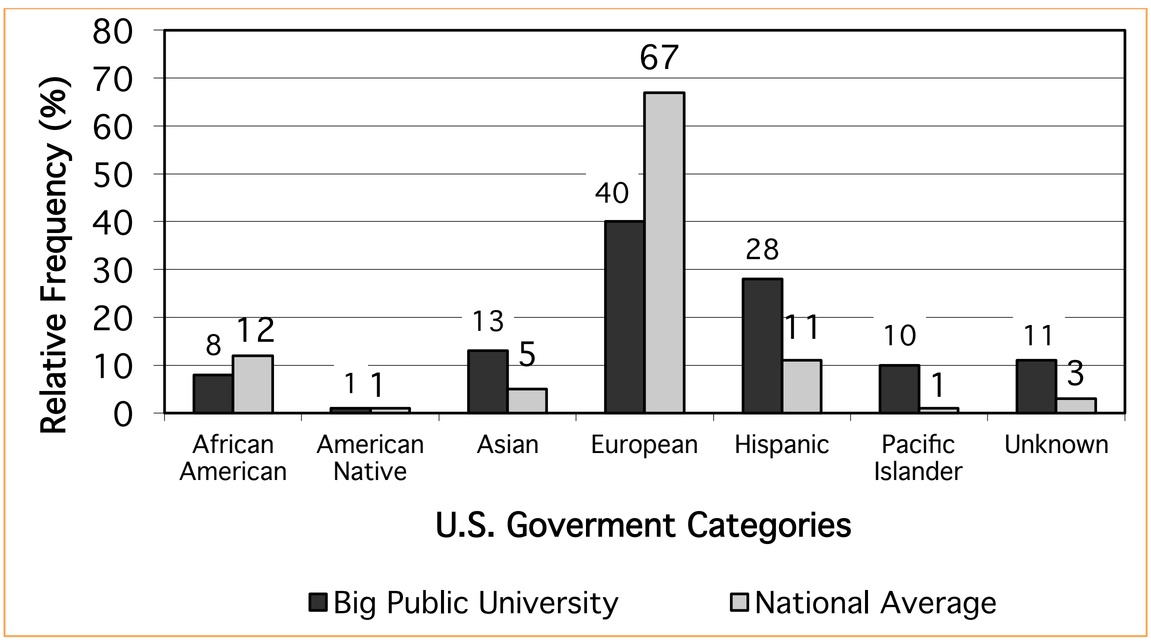


Figure 1. Student demographics at BPU compared to U.S. national averages.

Methods: Data Gathering and Analysis

Achievement

We collected algebra pre- and post-test scores, student preparedness information (SAT-Math and SAT-Verbal scores), demographic information, and course completion information.

All students in the 19 classes took a 25-item multiple-choice paper-and-pencil test over college algebra content in the first and last weeks of the term. The same test was used both times. Developed and reviewed by the instructor who coordinated the course and five expert college mathematics instructors, the exam was pilot tested in the year before being used for this study.

Assignments

The college algebra problem library programmed into WeBWorK for the study was made up of exercises selected from the textbook used by all the classes (Stewart, Redlin, & Watson, 2000; permission was obtained from the author and publisher).

The college algebra course coordinator determined a list of suggested homework exercises, organized by textbook section, and provided it to all instructors and to the WeBWorK problem library programming team. Each WBH and PPH instructor used at least 80% of these problems in weekly assignments.

Students completed the majority of homework outside of class time. Students in WBH courses did their WeBWorK on a home computer or at a computer in an on-campus lab.

Perception

At the end of the semester, WBH students completed a short survey designed to measure their comfort with computers and their perceptions of learning mathematics using the WeBWorK system. The survey included six statements, each with a five-point response scale, the seventh item was a prompt for written comments about WeBWorK. A similar survey of instructors was administered.

Results

Achievement

First, we give some context:

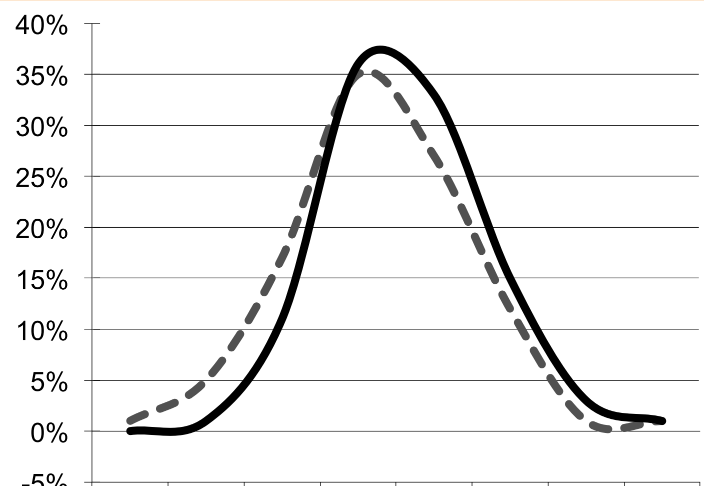
- *In the PPH classes, instructors reported that about 65% of students turned in homework regularly.*

- *The WeBWorK server records indicated that 78% of WBH students regularly attempted their web-based homework.*

The main statistical result was that no significant differences in performance were found between WBH and PPH students on the post-test nor were there any statistically significant differences in score gain between the groups from pre- to post-test (see Figure 2).

It seems WBH supports student achievement at least as well as PPH while saving instructors homework grading time.

We note here that, though disheartening, trends similar to those found in student achievement in high school algebra were present at BPU (e.g., with some students' score *gain* being negative or zero).



	-15	-10	-5	0	5	10	15	20
Paper&Pencil	1%	5%	17%	35%	27%	12%	1%	1%
WebWorK	0%	1%	11%	36%	33%	15%	3%	1%

Figure 2. Student score *gains* from pre- to post-test (25 points possible).

Student Perceptions

Student answers to the items on the end-of-term survey indicated that WeBWorK was seen as accessible and that they studied "about the same" amount with WBH as they had in previous PPH courses. Most students reported that they were already comfortable using computers when starting college algebra.

On the open-ended survey question, 149 students (of the 348 who completed the survey) offered written comments. We grouped them into three categories:

- *perceptions*
- *intentions*
- *belief-conflicts.*

The ethnic, gender, and course instructor distributions for the 149 responders were approximately those of the entire WBH population, though the distribution of grades was not the same as the whole population (students who ended the course with a grade of F were underrepresented in the 149 who made comments). Among the 149 responders, 40% perceived WeBWorK as "difficult to communicate with," noting: "Sometimes my correct answers would come up 'incorrect' because I did not type my answers the way the computer could understand." A small group of students (10%) also mentioned an urge to "put off homework because it's so frustrating" to use WeBWorK.

Student Intentions

As a support for engaging in mathematical thinking, WeBWorK is involved only as a monitor for correctness. Good monitoring is key in learning to be an effective problem solver. In the language of Schoenfeld (1992), the web tool does some monitoring but responsibility for meta-cognitive control (response to the monitoring), problem-solving, and the impact of mathematical beliefs rests on the student.

For the 35% of students responding whose comments indicated a view of mathematics learning as a complex and personal process of building conceptual understanding, WeBWorK was a tool that helped or hindered concept learning. The other 65% of students, whose reports indicated a procedural view of mathematics learning as a disconnected collection of formulae and "plug-and-chug" strategies, appeared to view WeBWorK as either helping or hindering a procedural approach.

Student Beliefs and Belief Conflicts

Student beliefs about mathematics appeared to be challenged frequently by their WeBWorK experiences. Spangler (1992) summarized **four main beliefs** about mathematics widely held by high school and college students:

1) *Mathematics is computation*; it does not involve reflection during task engagement;

2) *Mathematics must be done quickly*, or, spending little time is a more important task goal than sense-making;

3) *Mathematics problems have one right answer* and no further action or evaluation is required once an answer is found; and

4) *The teacher is the agent of mathematical learning, not the student* (i.e., only intentional acts on the part of the teacher lead to learning, no intention on the part of the student is necessary).

Many of the concerns voiced in student comments about WeBWorK can be traced back to a **violation of, or challenge to, one of these four beliefs**.

- As an illustration, in WeBWorK some computations can be done by the program. For example, given the problem: Solve for *x*: 3*x* + 1 = 7, a student who submitted, through the WeBWorK interface, (7-1)/3 would get back the response: "That answer is CORRECT." Some students reported feeling that they "weren't really doing math" because the program, not the student, would do such computation, **a challenge to Belief #1**.

- **Challenges to Belief #2** were evidenced in student comments about the role of time in using WeBWorK. Students could (and often did) retry problems. About 10% of students perceived a "re-try-ability" of problems that ***they said led them to further effort***. Another 10% commented with a tone more of complaint than self-reflection that they spent more time on their efforts in WBH than in previous PPH coursework and that *"math homework shouldn't take so long."*

- **Belief #3,** that mathematics problems have only one correct answer, appeared to conflict with the use of WeBWorK in two ways:

  First, WeBWorK would do computation for students so that (7-1)/3, 6/3, and 2 were all correct answers to the problem "Solve for x: 3x + 1 = 7." The possibility of multiple correct versions of an answer was a concern in about 10% of the student comments.

Secondly, and perhaps more significantly, were the reports by about 10% of student respondents that the goal was seeing "That answer is CORRECT." This group appeared to have both *the view that mathematics was a collection of algorithms* and the intention to aim for "that feeling of satisfaction" resulting from "That answer is CORRECT." Within this group there were four students who remarked on *guessing* many times when the first answer was not correct. We reviewed the WeBWorK audit trail and found that a very few students submitted as many as 35 guesses before moving on to the next problem. This small subset of students may not see their role as learners to include monitoring and control, so the monitoring offered by WeBWorK was of little use.

- **Belief #4,** came into play for the small group of students **who wanted WeBWorK to "be the teacher.**" About 15% of students said they disliked the fact that all they saw was "That answer is INCORRECT," and wanted "hints about what is wrong." WeBWorK may have been seen as a surrogate teacher *failing to be active because the interface did not suggest solution paths or give hints for how to proceed.*

Instructor Perceptions

The fifteen instructors came had a variety of mathematics and teaching backgrounds. See Table 1 for information on the instructors and their full-time equivalent (FTE) teaching experience (all names are pseudonyms).

WBH instructors also held differing view about the usefulness of WeBWorK.
As has been noted in the literature, what and how teachers communicate with students about innovation can impact its effects. Indeed, what instructors said about it was reflected in their student's survey comments and pre- to post-test gains. Table 1 on the next page summarizes the data for the instructors.

- Ms. Cone, Mr. Ellipse, and Mr. Graphic, each said in one way or another that they saw web-based homework as "not much use." This was reflected in their students' comments, including those who said it was "a colossal waste of time."

- On the other hand, Mr. Basis, Ms. Degree, Dr. Functional, and Ms. Join all said they thought WeBWorK was a good idea and "could be useful," but weren't sure it could replace regular homework. Each felt personal type of interaction was missing: they saw no way for themselves as teachers to guide students when the students made mistakes (connected, perhaps, to their awareness of students' tendency towards Spangler's Belief #4).

Students of these four instructors reflected their teachers' hesitant views of the usefulness of WeBWorK and included comments like "I prefer getting feedback from the professor because he could help me understand what I did wrong much better."

- Mr. Angle, Mr. Helix, Mr. Inch, and Ms. Kite all asserted that WeBWorK was a valuable tool and this was reflected in student comments about how "helpful" it was. Moreover their students, like those of the instructors in the "could be useful" group, also made suggestions for how the interface might be improved.

Table 1. Summary Profile of WBH and PPH Class Instructors.

WBH only	Degree at time of study	Years of teaching	Years teaching College Algebra
Ms. Degree	M.S.	> 10	> 5
Mr. Ellipse	M.S.	> 10	> 5
Dr. Functional	Ph.D.	> 10	3-5
Mr. Graphic	M.S.	> 5	3-5
Mr. Helix	M.S.	3-5	3-5
Mr. Inch	GTA	3-5	3-5
Ms. Join	GTA	< 1	< 1
Ms. Kite	GTA	< 1	< 1
PPH only			
Dr. Radian	PhD	> 10	> 5
Mr. Saddle	M.S.	> 10	> 5
Ms. Torus	M.S.	> 10	> 5
Mr. Undo	M.S.	1-3	1-3
WBH & PPH			
Mr. Angle (1W, 1P)	M.S.	3-5	3-5
Mr. Basis (2W, 1P)	M.S.	3-5	1-3
Ms. Cone (1W, 1P)	GTA	< 1	<1

Figure 3 below shows the average gain score for each instructor's students, with instructors grouped according to the opinion they expressed about the usefulness of WeBWorK. Note that though the initial assignment to WBH or PPH for each section was random; instructors had the choice to withdraw from either group. Two instructors switched from PPH to WBH; no WBH course instructor requested to be in the PPH group.

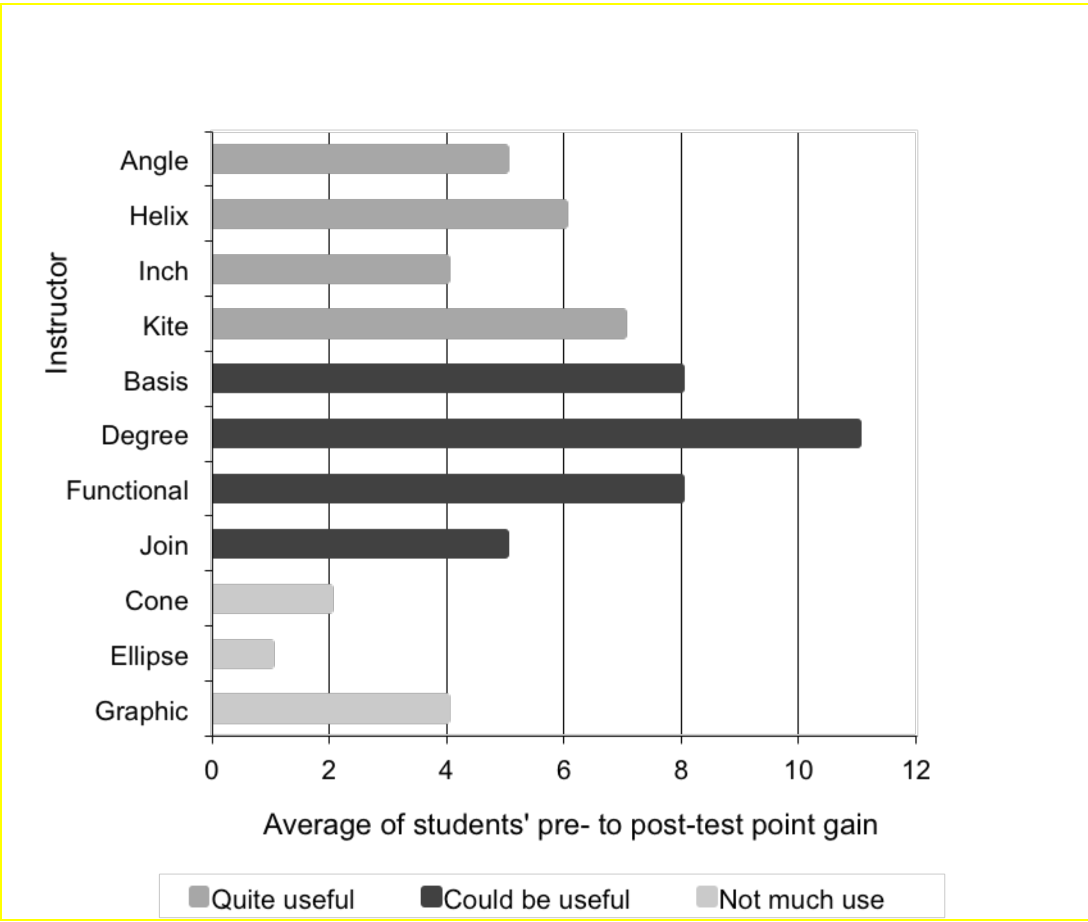


Figure 3. Instructors' views of the usefulness of WeBWorK and their students' pre- to post-test gains (out of 25 points possible).

Though the number of WBH instructors was too small to look for statistically significant differences among the performances of their classes based on a grouping by the instructor's perceptions about the usefulness of WeBWorK, the pattern apparent in Figure 3 is provocative. Certainly, when a teacher did not view it as valuable, student learning was prone to suffer by comparison (e.g., the bottom three bars for the "not useful" group in Figure 3).

Notably, the instructors who expressed interested hesitancy about the use of WeBWorK had higher average gains in their classes than those instructors who asserted they found WeBWorK quite useful.

The “could be useful” group of instructors reported carefully reflecting on what might be missed through the use of WeBWorK – qualitative feedback to their students – and said they implemented alternative methods for interacting with students. In fact, Ms. Degree (the instructor with the most experience, 21 years) assigned both WeBWorK and a few additional paper and pencil homework problems in her section. She carefully commented on these extra, mildly non-routine problems, before returning papers to students. Her WBH class also had the highest average gain from pre- to post-test.

Benefits and Limits of WeBWorK

Unlike internet auto-tutorials or discovery learning the web-based homework of WeBWorK investigated here does not openly conflict with traditional direct instruction or lecture methods of classroom teaching nor does it take a large amount of instructor time. This may be both good and bad.

- It is good in that the likelihood of WBH adoption by experienced teachers is increased because WeBWorK can be seen as a tool to reduce the need to grade piles of mathematics homework papers.

- It may be bad, however, in that WeBWorK does nothing explicitly to challenge the notion widely held by many students (and some teachers) that learning, particularly in algebra, is a matter of skill practice rather than construction of personal knowledge structures rich in conceptual connections to previous learning.

While it would be wonderful if WBH actually improved student performance, we think that an interface as straightforward as WeBWorK is unlikely to lead to such a result without additional teaching efforts (such as used by Ms. Degree). Nonetheless, WeBWorK can be used by teachers to make their teaching load more manageable while being at least as effective as PPH homework for most students.

A benefit of delegating the masses of skill practice for which PPH is viewed useful to a web-based interface is that it frees up instructor time and allows instructor choice in the nature of written interaction with students. That is, WeBWorK creates flexibility to spend what would have been homework grading time on alternative forms of feedback that may be more beneficial to both instructor and students (Cooper, Lindsay, Nye, & Greathouse, 1998).

Note:

This material is based upon work reported earlier (Hauk & Segalla, 2005). The work was supported by the National Science Foundation under Grant Nos. DUE0088835, and DGE0203225 and the U.S. Department of Education, Fund for the Improvement of Post-Secondary Education Grant No. P116B060180. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation or the U.S. Department of Education.

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MathFest 2009
August 6-8
Portland, Oregon

Mathematical Association of America

My First MathFest: What an Exciting and Rejuvenating Experience. And a Surprise![3]

Jonathan Brown

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MathFest 2009 in Portland Oregon was a three-day event from August 6th to August 9th. I had never paid attention to the MathFest Conference until I participated in two math competitions - the National Collegiate Problem Solving Competition sponsored by American Mathematical Society (AMS) and, Math Jeopardy, sponsored by Mathematical Association of America (MAA).

How did I become aware of these competitions? We have a math conference room and study lounge in the Math Department where posters, flyers, and information on such opportunities for math majors are routinely posted. A new problem is posted each month and I began submitting my solutions to these problems on a regular basis.

I found the problems challenging, but approachable and fun. In Spring of 2009 the math Department Chair, Dr. Abram, informed me that I had been named the “champion from our campus” as I had the highest number of correct solutions at CSU Stanislaus. I was elated, but the good news did not end there. Dr. Abram also told me that I was invited to attend the National Collegiate Problem Solving Competition at MathFest as our campus representative. This was all very exciting to me.

I researched MathFest and what I needed to do to travel to Oregon to prepare for the Competition in August 2009. For a student, especially for an undergraduate math major like me, obtaining adequate funds for this trip was a problem. I tried contacting MathFest about the issue, but did not get any real response. Fortunately, ‘lady luck’ was shining on my efforts. While wandering the halls of the math department over summer I ran into Dr. Sundar, Professor of Mathematics, who is delightfully enthused about mathematics and future mathematicians.

[3] Logo source: http://convention2.allacademic.com/one/maa/maa09/

She told me that MathFest would be such a great opportunity for my future endeavors and I should just plan my trip. Dr. Sundar assured me that if I didn't hear from MathFest or the AMS/MAA she would help me get there.

I immediately went home and collected math books that related to the problems I had encountered in the competition problems on the AMS/MAA board during the school year and started reading the brochures for MathFest. Looking at the conference schedule I realized how big MathFest really was. I was amazed at the number of math sessions that were scheduled everyday and all day! It seemed each session was by some of the biggest names in mathematics and on fascinating topics. While planning for the conference, I realized that there would be a Math Jeopardy event. I wanted to compete in this event.

Driving was the best and cheapest option for me. My wife and I packed our bags into the car and got off at about midnight. My wife drove, which helped me to get there on time as well as to get some sleep. The trip took about 12 hours. We washed up at a KOA (Kampgrounds of America) site outside of Portland, pulling into Portland the morning of August 5th.

I was greatly disappointed, however, when I learned that Jeopardy required teams of four. This meant that I could only register to watch the contest, but not to compete. In hindsight the confusion about registration was a great blessing. If I had known I wouldn't be able to compete in Jeopardy I would have planned to arrive the 8th and leave the 9th, and missed all of MathFest.

Now about the MathFest! Each day started early with the MAA Invited Addresses. Alan Taylor, of Union College, gave a talk titled Arbitrary Values of Arbitrary Functions. It was a fascinating talk, which left me daydreaming for months about cutting cakes democratically and guessing hat colors. Democratic cake cutting is a variant of the "fair division problem", which is to create an algorithm for n agents to divide a resource with some amount of fairness without an arbiter, and "guessing hat colors" refers to the task of successfully guessing hat colors based on the colors of the hats in front of you in a line.

Ravi Vakil, introduced as the "Rock Star" of mathematics, was amazing at clearly describing complex structure in mathematics. Through the analogy of a common doodle Vakil illustrated the limit of a sequence of functions under composition and the speed of convergence. The results were interesting for anyone who has studied the theory of functions of real variables and yet still a pleasure to follow for someone who has not. It would be just as futile to try to enumerate his accomplishments, as it would be to remove the seeds he planted in my head that have continued to keep me wondering about, as he put it, "the way the universe wants us to see things". Each of his three talks in the series entitled "Modern Structure in Classical Geometry" was well worth the trip.

Beside the MAA Invited Addresses there were 'paper sessions,' many five hours long. A paper session is a collection of people giving talks about papers they wrote on a common topic. Of the paper sessions the three I found most pleasing were "Open and Accessible Problems in Knot Theory", "Graphs, Networks, and Inverse Problems", and "History of Mathematics".

It wasn't only the invited addresses and paper sessions that made MathFest awesome- it was the people. I was very happy to see Mrs. Doan, a professor from Victor Valley Junior College, who was presenting a fascinating paper that revealed the lost history of the longitude problem. She treated my wife and I to lunch, which gave us an opportunity to catch up and discuss common interests.

One night there was an ice cream social designed to bring people out and honor some undergraduates who had won awards for their research. I decided to bring a board game, Settlers of Catan, with me to the ice cream social and I got to play and network with some newly graduated Ph.D.'s. Just listening to them talk about their research and experiences made me feel confident about my own preparation for the road ahead of me- which is to enroll in a Ph.D. program.

Although I did not win the Problem Solving Competition, some of the judges and people in the audience informed me that my performance was impressive. I was a little disappointed I didn't get a plaque to bring back to CSU Stanislaus.

I want to say that I could not have done this without the support of the CSUS Math Department, which routinely posted the math problems for the interested students, Professor Sundar's vision for future mathematicians who generously funded the trip, and my non-mathematician wife. There is no way I can adequately thank her for all she did to help MathFest be a great experience for me and who also sat through three days filled with my non-stop talks about mathematics. We were both revitalized by the trip to MathFest in beautiful Oregon.

I strongly recommend to fellow students that if you like doing math problems or have a desire to pursue a graduate degree in math, do attend MathFest. (See: www.maa.org/mathfest/)

English Learners in the mathematics classroom, a view from critical hermeneutics.

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Every view of the world that becomes extinct, every culture that disappears, diminishes a possibility of life.
Octavio Paz

Introduction

In the K-12 classrooms of a society that readily accepts and rather enjoys the cultural customs and cuisines of its diverse citizens but is slow or unwilling to accept mutatis mutandis the uniqueness of their children's cultural learning styles diminishes the potential of those children.

The aim of this paper is to use the findings of the field of critical hermeneutics, which we shall define presently, to encourage teachers, in particular mathematics teachers to add to their existing "differentiated instruction" as defined in the California Mathematics Framework.

The essence of the field hermeneutics is the study of how cultural variables play an important part in the teaching-learning process. The reader is encouraged to read above and beyond this article to know how critical hermeneutics can help EL learners improve their learning.

Hermeneutics experts believe that educators have not taken the cultural aspect of the teaching-learning process to a sufficiently meaningful level in order to ensure that all students have the opportunity to show their academic potential. The underpinnings of this theory are the linguistic characteristics of a culture: Awareness of the linguistics differences one can better understand the cultural aspect.

As teachers, we have applied various pedagogies from the age-old lecture method to constructivism. Perhaps it is time to add to the aggregate of pedagogical practices the heretofore neglected cultural learning styles of students, to achieve what hermeneutics experts call "social justice" in the classroom.

The mathematics classroom is a natural place to apply the ideas of hermeneutics as mathematics is a discipline that transcends cultures. But more: we have not done very well with usual methods of instruction in teaching young children.

When we compare United States children's mathematics achievement with that of other countries in the TIMSS[4] and PISA[5] studies, our children rank low. Of course the evaluation has to be tempered because of our culturally heterogeneous student population as compared with that of other countries. But why not turn this difference into a strength by using the talent—for mathematics and other subjects—that is waiting to be discovered behind the "cultural curtain." We need to boost our determination to have *every* child meet her mathematics potential. After all, the Declaration of Independence of this great country states:

> *We hold these truths to be self-evident, that all men are created equal,*
> *that they are endowed by their Creator with certain unalienable Rights,*
> *that among these are Life, Liberty and the pursuit of Happiness.*[6]

Some facts

- There are about five million "English Learners" in this nation's K-12 classrooms, representing almost 100 different languages.

- A majority of elementary school English Learners (EL) are native to the United States.

- California educates about one-third of the nation's estimated EL students: 1.6 million students.

- Eighty-five percent of all EL students in California are Spanish speaking.

These facts and figures are from 2007, and have been subjected to slight adjustments to the references below:

- Until 2009-10 when Hispanics/Latinos edged over 50% in California, the state had no majority ethnic group.

- The state has larger percentages of Hispanics/Latinos, Asians, and American Indians than other four large populous states like New York, Texas, and Florida.

- In 2007-08, Hispanics made up more than 48% of California's student population as compared with 21% nationally. (NCES)

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[4] Trends in International Mathematics and Science Study (TIMSS) compares mathematics and science achievement of 4th and 8th grade students in the US with that of other countries.
[5] Programme for International Student Assessment (PISA) measures mathematics skills necessary for productive citizenship; it is administered internationally to15-year-olds.
[6] The word "unalienable" appears in the Declaration of Independence, but often it is replaced in conversation and writing by its synonym "inalienable."

Hermeneutics defined and its relevance to education

The Stanford University Encyclopedia of Philosophy defines the term hermeneutics as *"the theory of understanding and interpretation of linguistic and non-linguistic expressions. As a theory of interpretation, the hermeneutic tradition stretches all the way back to ancient Greek philosophy and includes the study of the classical ancient cultures*."

In the classroom we know all to well that a child must understand the language in order to learn, but the interpretation part of the definition is as important. English is a rich and expressive language with as many innuendos and connotations as one would want, which is good for Shakespeare, but not good for the typical EL student. As teachers we must be aware of the richness of the English language, and make every effort to have our students interpret the language in a word problem correctly—a crucial part of problem solving.

Although this is not the place to pursue the history of hermeneutics all the way back to the Greeks, as suggested by the definition above, one example stands out: Plato's Meno includes a discussion of how Socrates evokes from a young man the knowledge to double the area of a square. The entire discussion in the Meno revolves around the importance of interpretation as suggested by the definition of hermeneutics above.

Hermeneutics' theme has been present since Aristotle. This does not mean that we should still follow the teachings of Aristotle—at least not literally—but certainly in spirit. For the purpose of this article let us simply take one key word: *culture*. We must recognize that in the K-12 classroom, all our students wear some cloak of enculturation; and, as defined above, the culture's linguistic characteristics. Be they "American" or foreign born, the teacher can improve the learning, again, in this case let us say, mathematics, by paying more attention to vocabulary and study habits of students.

In a study done at the UC Berkeley by Uri Treisman (Studying students studying calculus: A look at the lives of minority mathematicians. 1992. *College Mathematics Journal, 23*, 362-372) researchers concluded that "many minority students, especially Black and Latino students, did not use the services that were designed to help them. One of the reasons for this was identified as being cultural." If culture plays a role in the teaching-learning process at Berkeley, it certainly can happen in our own classrooms.

Vocabulary

The vocabulary in mathematics is rather unique and since (too) many people are apprehensive about the subject its symbolic language and unique vocabulary add to its undeserved reputation for impenetrability.

Here are just some of the many words and phrases that add to the mystery of the subject and play havoc with EL students:

> *quotient, difference, product, divide (into two parts, not perform the division; or divide by as oppose to into), GCF, LCM, factor, separate, combine, at least, at most, no more than, not less than.*

Critical hermeneutics theory would suggest giving EL students an appropriate "dictionary" list of mathematical terms and possibly interpretations along with "translated" words and phrases. This author believes that it will not be long until EL students "lean over" to using the regular, accepted vocabulary. Along with this type of activity, critical hermeneutics would suggest using diagrams, graphs, figures, stick people drawings, and other visuals to help EL students interpret the connotations in the English language. This is especially true for the dreaded word problems, an area of mathematics where most students have difficulty.

Pre-understanding

"Understanding is not possible without pre-understanding" is a major theme in critical hermeneutics: students in mathematics classes, EL students especially, need to have a springboard of "pre-understanding" before they delve into trying to understand the quintessence of a mathematical concept. Here is an example.

> *Ellen bikes to school every day from her house. Today when Ellen was three quarters of the way to school she met Esther who was walking to school. They decide to walk the rest of the one half mile to school together. How far is the school from Ellen's house?*

"Bikes" could be misinterpreted as a noun, not a verb, as it is used in this problem. Perhaps "*three quarters*" needs to be restated in fraction notation for better EL understanding. "*The rest of the one half mile . . .* " is also a key phrase, for what is the meaning of "the rest" to an EL students? As a matter of fact there are no numerals or symbolic fractions in the statement of the problem at all. The EL students must interpret all these facts using language skills and interpret them correctly.

Critical hermeneutics replaces "how to read" a problem like the one above with "how do we communicate the ideas in the problem?"

Solutions of mathematical problems cannot exist without communication with other people or even with oneself. Yet, teaching to encourage "equal voices in the classroom" can be an alluring experience for children. Such equal voices can transform present teaching practices to the point that would bring us to a true state of equity.

This author's experience with critical hermeneutics—teaching for social justice as we stated above—in the classroom has had positive results. It is her belief that specific hermeneutics research in mathematics education will more than likely reveal how to improve mathematics education especially for EL students.

We encourage teachers to follow a methodology that has been shown to improve the teaching-learning process. We especially need this in the mathematics education classroom. How? The hermeneutics teacher encourages and inspires students to speak "the dialogue in which we are" and to "bring cultures together," in the classroom.

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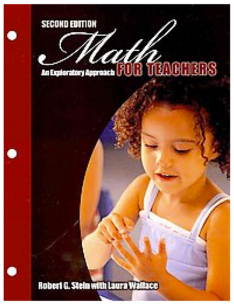

Math for Teachers: An Exploratory Approach

Robert G. Stein with Laura Wallace (Authors)
Publisher: Kendall Hunt (2010)
Reviewed by Mark Bollman, on 03/04/2011

Reprinted from MAA Review

At my current institution, "Mathematics for Elementary Teachers" is a one-semester course that meets for 6 hours per week, ostensibly divided into three hours of lecture and three of laboratory weekly. I have been teaching that course for many years — indeed, no one else currently in my department has taught it — and in that time, I have looked at many textbooks for that audience and that course. As is the case with many standard service courses, there seems to be considerable agreement on most of the topics to be covered, so I have developed my own core list of criteria for evaluating these books.

First, I hope that "A Math-For-Elementary-Teachers" textbook will be a resource for future teachers — something they can keep with them as they move out of my class and into their first teaching position.

On that score, Stein and Wallace have written a fine text. The emphasis is on the mathematics, and while the students' goal to teach is not far from the surface, the content manages to dominate. Indeed, there is no laundry list of NCTM Standards to detract from the primacy of the mathematics. (I accept that others may regard this as a flaw.)

I also hope that students will find the mathematics they will use as professionals in their textbook, and so I look carefully for a full section explaining the normal distribution and the mathematics behind percentiles, which teachers will need when trying to interpret their students' standardized test results.

Unfortunately, no such section is present here, though there is a very brief mention of percentiles. While that is a flaw in my opinion, it's one that can be easily filled in by those who feel it's important.

That, however, is the only concern I have about this book. The standard topics are all here and covered in an unusual level of detail — which is to be expected when the book includes more than a year's worth of material. A student armed with this book and with the experience of learning from it will be well-prepared, mathematically, for a career as an elementary school teacher.

Mark Bollman (mbollman@albion.edu) is associate professor of mathematics at Albion College in Michigan. His mathematical interests include number theory, probability, and geometry. His claim to be the only Project NExT fellow (Forest dot, 2002) who has taught both English composition and organic chemistry to college students has not, to his knowledge, been successfully contradicted. If it ever is, he is sure that his experience teaching introductory geology will break the deadlock.

This review is reprinted from MAA Reviews http://www.maa.org/maareviews

Teaching Problem Solving at the Elementary Level Using A Deck Of Playing Cards

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Elementary school classrooms seldom use playing cards in the teaching of mathematics, probably because of the gambling stigma that has long, and justifiably, been associated with playing cards and gambling. But times change, and petty correlations to iniquitous activities relegated to such objects as playing cards, dice, and gambling in general, has weakened to the point that we may use some of these games as instructional aids and manipulatives. In short, using playing cards as a manipulative in the classroom is, in our view, more than acceptable nowadays. It is desired.

That being said, playing cards can be used as manipulatives for illustrating mathematical concepts from numeration to probability and more. Elementary school children can grasp simple numeration problems using 'playing cards,' 'rolling dice (or one die)' or the lotto type 'drawing a black ball from a jar that has black, white and red balls.'

This article attempts to show how a regular deck of playing cards, with an additional "zero card" or "joker" for each of the four suits may be used as a 'manipulative' to illustrate some concepts involving practice with whole number operations, number sentences, elementary geometry (similarity and congruence) and elementary algebra (evaluating variables).

For the reasons outlined above our experience show that children tend to welcome a game using a 'regular deck of cards' more than 'typical flash cards.' Why? One, playing cards is a fresh manipulative, and two, because it does still have the sense that cards are used as an adult past time. Undeniably our school children are well aware of the ultra adult televised versions of poker and blackjack contests, though we would wish that they would be as savvy of the more tactical game of bridge.

Below is a brief description of some mathematical concepts that can be taught and reinforced using a deck of playing cards as a manipulative. All the games can be played in small groups of students or with the entire class. The teacher needs to keep in mind that the numbers is not as important as its conceptual representation. Once the form of the game is well established, several variations can be introduced.

The games are arranged on the following pages so that they may be easily duplicated.

Sorting Game

- **Materials needed:**
  - Any one of the four suits of 13 cards and
  - “a zero card” (or Joker) for each participating student.

- **Objective: to have students be able to**
  - recognize symbols and numerals;
  - associate numbers with variables;
  - arrange the cards in ascending or descending order.

- **Directions:**
  - Sort out the cards into the 4 suits of 13 cards each and one Joker; give each student one suit and have the students shuffle their cards. Instruct the students to hold out the card that you call for.
  - Call out for ‘2’ or hold out ‘2’ written on a plain piece of paper or write ‘2’ on the board.
  - Ask all students to show their ‘2’ card.
  - Repeat this for other numerals; Jack, Queen, King, and Joker (0).
  - Next, ask all students holding Jack of hearts to stand up (or scratch their ears).
  - Repeat this with other cards. This game may be played in groups of 2-6 where you select a student to act as teacher.

Missing Card Game

- **Materials Needed:**
  - Regular deck of playing cards.

- **Objective- to have students be able to**
  - recognize the cards;
  - sharpen memory
  - identify the symbols on the card
  - use the correct language.

- **Directions:**
  - Start with any six cards.
  - Place them face up on a mat.
  - Have the student(s) study the cards.
  - Tell the student(s) to look at the ceiling.
  - Remove one card.
  - Ask one of the children to name the missing card.
  - Repeat.
  - Again, ask the same or another student(s) to look away and remove another card.
  - Ask the student(s) to name both of the missing cards.
  - Remove a different card with or without replacing the first two.
  - Continue until the student(s) has mastered that particular set of cards.
  - Slowly increase both the number of cards in the play as well as the number of cards removed at each step.

Memory Game

- **Materials Needed**
    - 2 – 4 decks of playing cards depending on the number of players.
    - At least 4 cards per player.

- **Objective: to have students be able to**
    - match shapes, size and color
    - develop the concept of one–to-one correspondence
    - compare size of numbers
    - count and skip-count by two

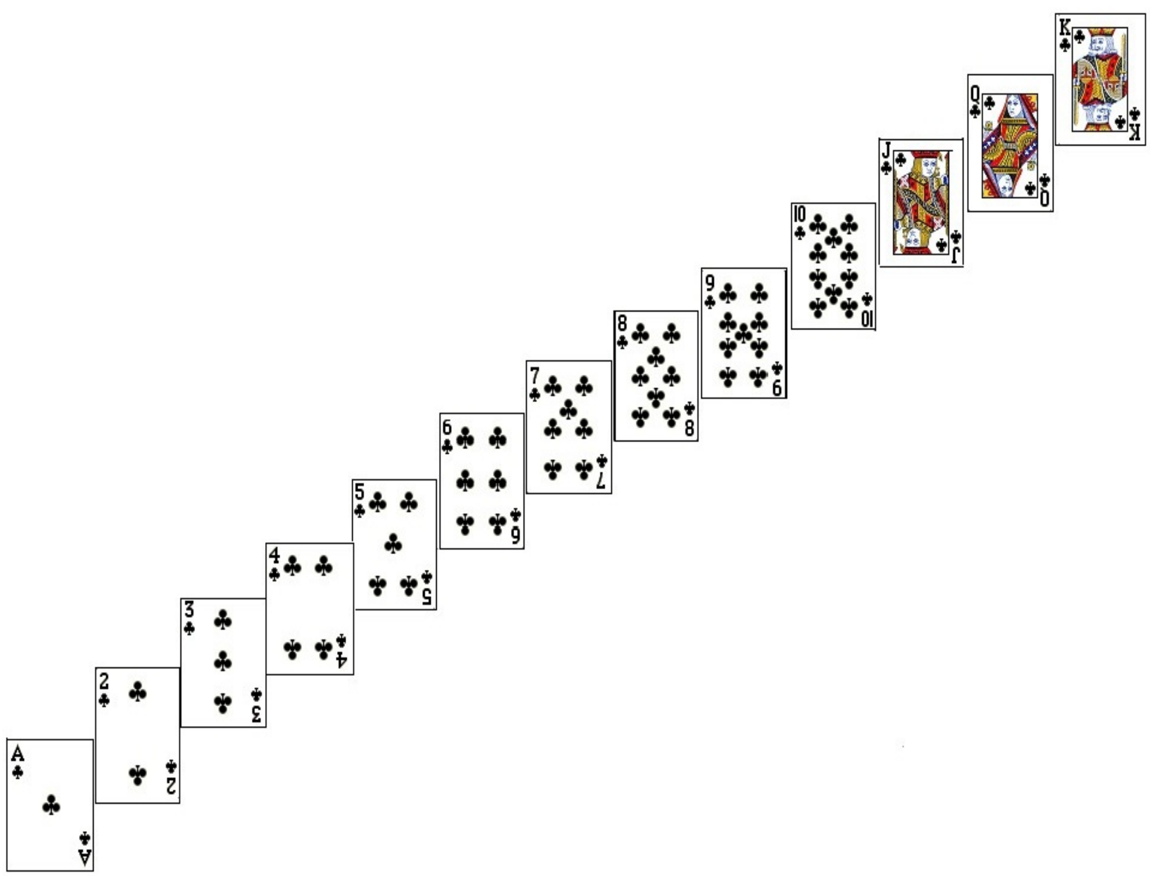
- **Directions;**
    - First decide whether you want to match shapes (for example, any two hearts or any two diamonds) or numerals (the 2 of diamonds matches the 2 of hearts), etc.
    - For younger groups begin the game with only two suits of the same color.
    - Shuffle cards and place all cards face down.
    - Each player is allowed to turn two cards face up. If the cards match, the player keeps them and gets an extra turn. If the cards don't match, the player to the left takes the turn.
    - Play continues until all sets are formed.
    - Each child counts the total number of sets and the total number of cards. The teacher records them. They compare scores; observe the doubling pattern between number of sets and number of cards.

Staircase Game Up

- **Material Needed**
  - Deck of cards, one suit of 13 cards and a zero (Joker) card for each student

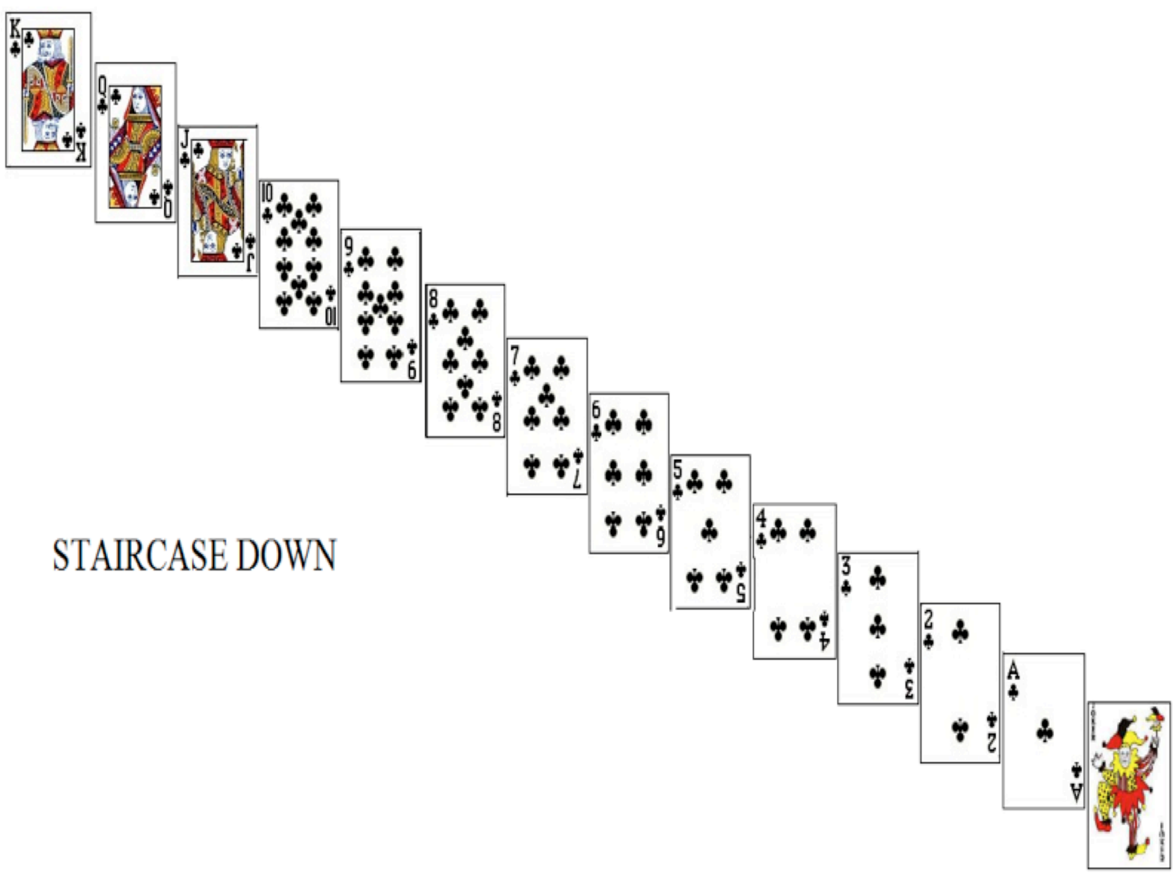
- **Objective: to have students be able to**
  - recognize symbols and numerals
  - arrange the cards in ascending order.

- **Directions:**
  - Give each student one suit of suit of cards – Ace through King and the Joker or a zero card.
  - Instruct the students that Joker = 0, A= 1, J= 11, Q=12, and K=13.
  - Have each student shuffle the cards, lay them face up on the mat and rear-range them in the ascending order.



Staircase Game Down

- **Material Needed**
  - Deck of cards, one suit of 13 cards and a zero card for each student
- **Objective: to have students be able to**
  - recognize symbols and numerals
  - arrange the cards in descending order
- **Directions**
  - Give each student one suit of suit of cards – Ace through King and the Joker or a zero card.
  - Instruct the students that Joker = 0, A= 1, J= 11, Q=12, and K=13.
  - Again have the student shuffle the cards, lay them face up on the mat and then rearrange the cards in the descending order starting with King.



A look at pedagogical implications

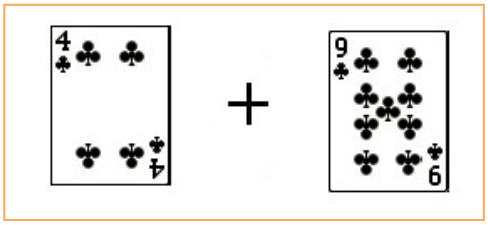
Below, I shall indicate some of the problems I asked an eight-year-old and the answers she presented. Her solutions to the last two problems were particularly interesting and unexpected. She did not illustrate division (÷) the way I expected her to do. However, she did present the correct answer *with a verbal explanation*. As is evident from the exercises here, these can be adapted to illustrate many different concepts involving practice with whole number operations and number sentences.

In most of the problems below we "go with" this interesting pedagogical insight, that is, the child's way of looking at numbers. Historically, this will remind us of the additive system the ancient Egyptians used; but not quite the same since these cards have a numerical value, while they are "pictorial. In ancient Egyptian numeration for example the symbol for the number ten was a heel mark ∩ and for one hundred a scroll, for one thousand a lotus flower, and so on. Here, the "pictures" on the numbered cards serve a dual purpose; the number eleven is symbolized by a Jack; twelve by a Queen; and thirteen by a King,

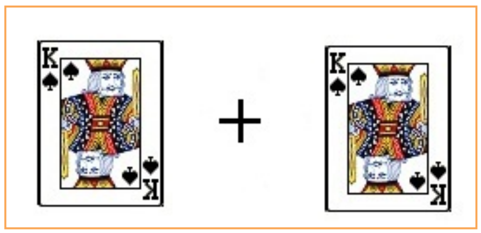
With this caveat, let us proceed to the questions in the pages below.

Word problems and the way they were answered:

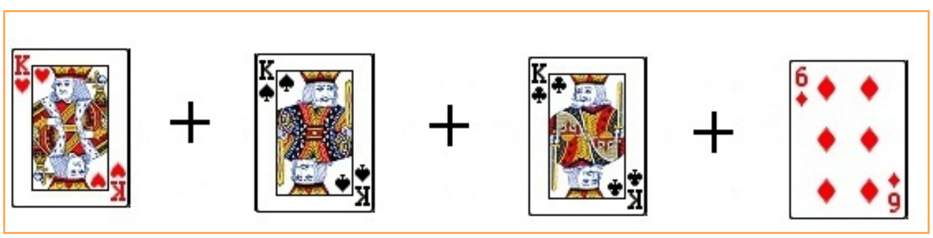
1. There are 30 students in your class of which 17 are girls. How many boys are there in your class? Show me the answer.



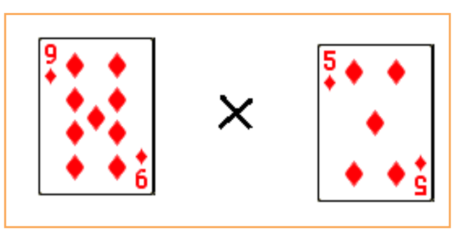
2. There are 30 students in your class. If four students are absent how many are present? Show me the answer.



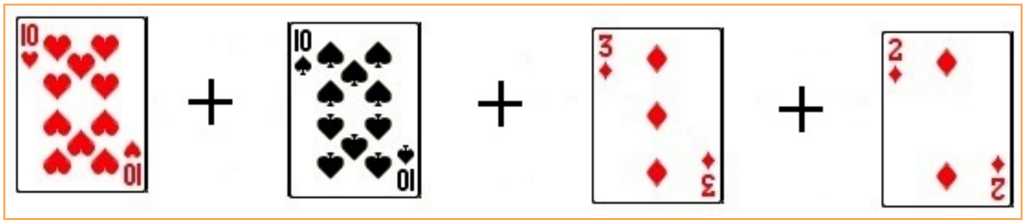
3. You bought a bubble gum for 10 cents, an eraser for 15 cents, and a balloon for 20 cents. How much will you have to pay in total? Show me the answer in two ways.



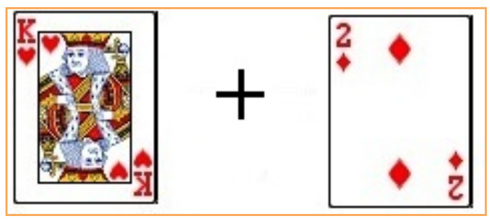
Or



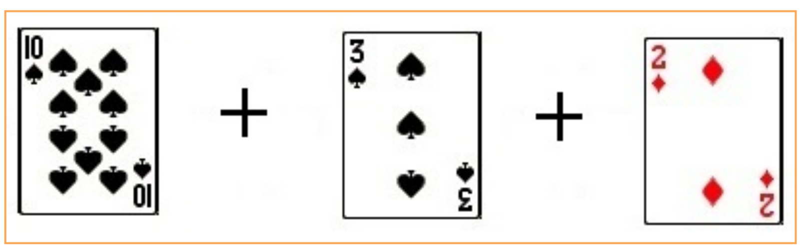
4. Suppose we are at the store to buy school clothes. I have $ 40. I buy you a pair of shoes for $17 and a shirt for $8. How much do I pay? And how much money is left over   Show me the answers.



Left over:



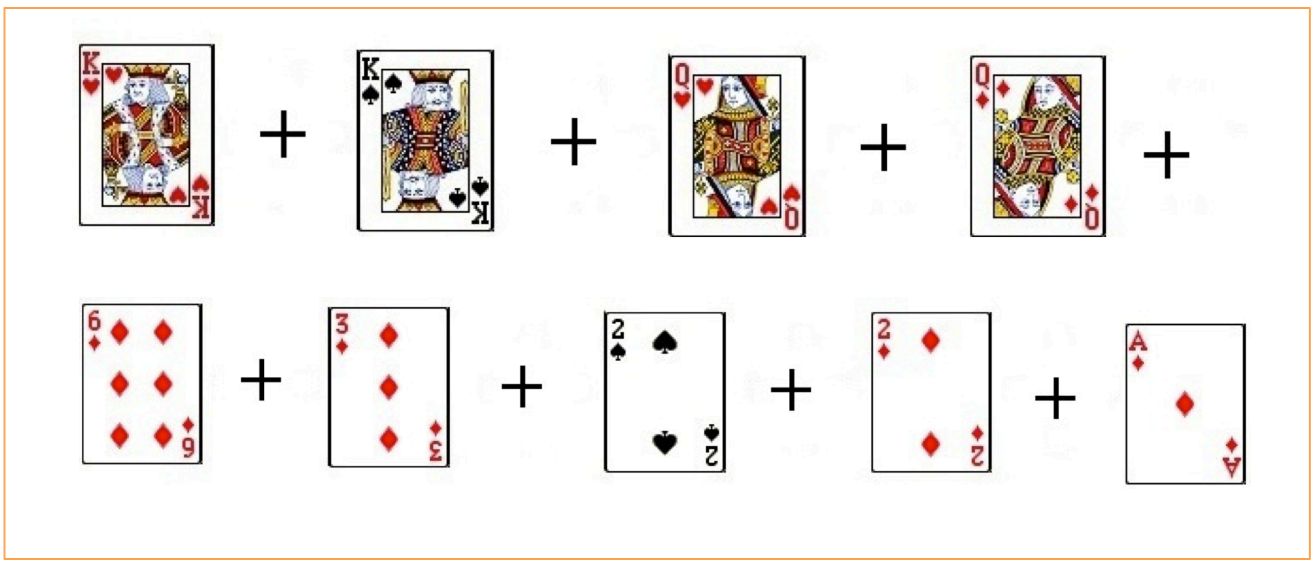
or



or

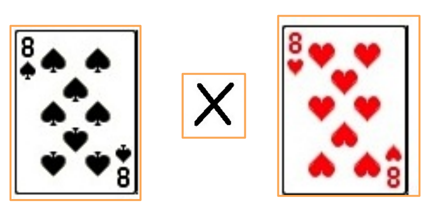
5. You have $1. Each pencil costs 9 cents. How much will 4 pencils cost you?

How much will you have left over?



Can you do this with fewer cards?

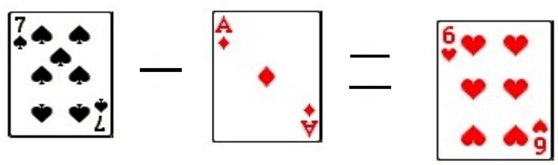
Or



6. Suppose you have 19 apples and you gave 2 apples to each child. To how many children can you give apples?

Nine children and 1 apple left over

7. You have $30 to buy gifts for your friends. You decide to buy belts. Each belt costs $5. How many belts can you buy?



What does it mean?

Comment: First I thought I could buy 7. But that didn't add up to 30.

(She did the problem by repeated subtraction and set up an equation to give the correct answer 6.)

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