Journal of the California Mathematics Project
Volume **11**, 2022

Using Pitfalls to Support Middle School Mathematical Discussion and Equity

Kathleen D'Silva, Elizabeth Dyer, and Jodi L. Davenport

Abstract. This study examined how three seventh-grade teachers implemented discussions of mathematical stumbles and errors (pitfalls) while using an instructional supplement. Using a "Math-Talk" framework as a lens, results indicate that some teachers faced a conflict between what they saw as important in maintaining student trust (e.g., validation of correct answers) and giving time and attention to pitfalls. One teacher who celebrated mistakes as learning opportunities, saw that discussion of pitfalls lead to more equitable student engagement. We examined the variation in facilitating discussion across teachers and offer a possible extension to the Math-Talk framework.

Theoretical Perspective and Background

Access and equity in mathematics classrooms require that all students "participate meaningfully in learning mathematics" (National Council of Teachers of Mathematics [NCTM], 2014). Many equitable teaching methods, such as focusing on mathematical reasoning and mathematical practices, involve the use of language (NCTM, 2014; Moschkovich, 2013). In an equitable classroom, all students participate in discourse. A mathematical discourse community is generated when the teacher and students agree to work on meaningful communication about mathematics (Willey et al., 2017).

Mathematical discourse "includes the purposeful exchange of ideas through classroom discussion, as well as through other forms of verbal, visual, and written communication" (NCTM, 2014). Mathematical discourse communities improve student problem solving and deepen conceptual learning (Hufferd-Ackles et al., 2004; Murata et al., 2017) while promoting equity (NCTM, 2014; Michaels et al., 2008). Walshaw and Anthony (2008) said "explanations stimulate, challenge, and extend other students' thinking" (p. 25), but caution that discussions only enrich classwork when all students are included.

Seemingly productive discussions can leave out some students or may not yield deep, conceptual learning (Murata et al., 2017; Walshaw & Anthony, 2008), thus it is important that all students participate in the discourse in meaningful ways. But how is that possible when students are not always "correct?" Focusing discussion on explaining wholly or partially incorrect problem-solving strategies, or pitfalls, may be a key strategy in creating effective conversations because it communicates that all ideas are valuable (Booth et al., 2013).

Barbieri and Booth (2016) found that when lower-performing students reflected on and explained incorrect solutions, or pitfalls, their algebra learning improved more than comparable students

who only examined correct solutions. When questions focus students' attention on common pitfalls, students must directly confront common misconceptions and are less likely to make similar errors in the future. Explanations related to pitfalls, can generate inclusive, productive mathematical discussion. When conversations acknowledge, confront, and unpack pitfalls, they support student sense-making (Booth et al., 2013). The current study explores how a classroom focused on sense-making around mathematical pitfalls might yield more effective and inclusive discussions.

The Math-Talk Learning Community Framework (Hufferd-Ackles et al., 2004) describes the developmental trajectory needed to create "math talk learning communities" where the primary goal is to "understand and extend one's own thinking as well as the thinking of others in the classroom" (p. 2). The framework emerged from a case study of one teacher who, over the course of a year, moved from a traditional pedagogy to using more reform-based practices for success in whole class discourse. The framework was then refined based on data from three other classrooms and has been used in numerous studies since its development (e.g., Murata et al., 2017).

The base level, Level 0, of the Math-Talk framework describes a traditional teacher-directed classroom while at the top level, Level 3, students have ownership of the classroom talk and the teacher is facilitator, co-constructing knowledge and discourse with students. Within the Math-Talk framework are four components: questioning, explaining mathematical thinking, source of mathematical ideas, and responsibility for learning. In particular, the *source of mathematical ideas* attends to who owns mathematical ideas, including who decides that an idea is mathematically valid.

See Table 1 (next page) for brief descriptions of each math-talk level for *source of mathematical ideas*. In the current study, we used the construct of *source of mathematical ideas* as an interpretive lens to explore the ways teachers viewed and used mathematical pitfalls for classroom discussion.

Methods

Research Questions. This paper focuses on two exploratory research questions: How do teachers make sense of the role student pitfalls play in class discussion? How might the handling of student pitfalls support/hinder equitable access to mathematics?

Approach. We used qualitative methods based on constant comparative coding of interview and observation data (Patton, 2015), first for themes, then for patterns, and finally for relationships and distinctions among patterns using the Math-Talk framework. Member checks were done by sending participants drafts of this manuscript and requesting feedback on the accuracy of the analysis.

Participants. All three case study teachers, Rita, Sean, and Jane (pseudonyms) taught seventh grade math at public middle schools in California. All had secondary mathematics credentials, at least six years teaching experience, and had taught at their schools for at least three years. Participating teachers engaged in a pre-implementation workshop with virtual follow-up sessions. Subsequent case study data collection was March to May 2018.

Table 1. Math-Talk Framework: Source of Mathematical Ideas (Hufferd-Ackles et al., 2004, pp.88-90).

Level 0	**Level 1**	**Level 2**	**Level 3**
Teacher is physically at the board, usually chalk in hand, telling and showing students how to do math.	*Teacher is still the main source of ideas, though she elicits some student ideas. Teacher does some probing to access student ideas.*	*Teacher follows up on explanations and builds on them by asking students to compare and contrast them. Teacher is comfortable using student errors as opportunities for learning.*	*Teacher allows for interruptions from students during her explanations; she lets students explain and "own" new strategies.* Teacher uses student ideas and methods as the basis for lessons or mini-extensions.*
Students respond to math presented by the teacher. They do not offer their own math ideas.	Some student ideas are raised in discussions, but are not explored.	Students exhibit confidence about their ideas and share their own thinking and strategies even if they are different from others. Student ideas sometimes guide the direction of the math lesson.	Students interject their ideas as the teacher or other students are teaching, confident that their ideas are valued. Students spontaneously compare and contrast and build on ideas. Student ideas form part of the content of many math lessons.

**Teacher is still engaged and deciding what is important to continue exploring.*

Setting. Teachers taught several two-day lessons from a supplemental curriculum focused on exploring student thinking, *Math Pathways and Pitfalls*® (MPP; Moorjani & Kao, 2019), as part of a larger quasi-experimental study. Lessons started with a class discussion of the lesson purpose and relevant "math words" (i.e., vocabulary), followed by a starter problem, meant for students to work on independently. Students then considered and discussed the printed work of two student characters: one correct and the other with a common pitfall. MPP is based on the principle that jointly examining correct work and confronting pitfalls supports mathematical discourse community development. During instruction, this is realized through reflection and exploration of the problem solving of the characters in the text and of students in the room. MPP provides teacher prompts and a poster of "discussion builders," or sentence stems to scaffold students' classroom conversation (e.g., "I have a question about [another student]'s idea. . . ").

Data Sources for the Research. Data collected included an initial interview, observations of two different two-day MPP lessons, and, after each lesson, an interview with the teacher. In the initial interview, we asked teachers their perspectives on how discussion related to equitable access to the mathematics (Part A) and strategies for building mathematical discussions (Part B). Observation data were collected as a running record with prompts to focus observers on equity and access with attention to both teacher and students. The first post-observation interview was similar to the initial interview but related the ideas to the observed lesson. The second post-observation interview focused on curricular tools that helped or hindered discussions as well as teaching dilemmas associated with discussions.

Results

In their interviews, teachers talked about the benefits and challenges of using pitfalls in classroom discussions. These reports and observations on discussion within the Math-Talk framework are given below.

Sean - Interview. Sean said students learned from the pitfalls in the curriculum, "knowing the answer is right or wrong, the kids have an easier time explaining why it is correct or explaining and checking why it is wrong." However, he said students were hesitant to share and learn from their mistakes. Sean said he did not ask students to present incorrect solutions if he had checked their work because he did not want to violate their trust. In the interview, Sean said the "beautifully-done MPP days, is when I talk the least. So, if yesterday I talked 55% of the time and today I talked 45% of the time, that is a successful day." He thought the MPP curriculum's focus on pitfalls supported student engagement. Also, Sean felt he encouraged engagement by giving students points for participating in discussions but limiting the maximum points they could earn to make space for less eager students to contribute. Engagement was further strengthened, he said, by giving students time to share in small groups before sharing with the full class. Sean was very aware of the percentage of students who spoke who were of different backgrounds (e.g., race, gender, and math ability) and academic habits. He was very focused on ensuring diversity of voice and found that the curricular focus on pitfalls, combined with his strategies increased engagement. However, he noted that students who did not normally participate (on non-MPP days) did not participate in the MPP lessons either. He desired more inclusive engagement.

Sean - Observation. In Sean's MPP lessons, the math-talk community demonstrated some Level 2 elements, with students sourcing some mathematical ideas. Sean followed the general format of MPP lessons, asking students to discuss worked examples and pitfalls in pairs, followed by students presenting their work to the class. He elevated the status of student voice by calling presenting students "maestro/a" and "professor," and facilitated students asking questions of each other. He asked students to respond to the presenter using the discussion builder sentence stems. Sean's prompts probed student thinking and he occasionally asked students to make connections between different problem-solving strategies. The community also demonstrated Level 0 and Level 1 elements, with Sean becoming the source of ideas when students struggled. When Sean probed students' thinking, students often did not respond, and Sean narrowed his questioning towards more short answer/recall prompts. If students did not volunteer to present or struggled, Sean explained, asking for student input. While Sean stated in class that student errors were valuable, errors were not used as a platform for learning. Neither teacher nor students probed thinking around errors made by either the fictional students or those in class. When students presented incorrect solutions, Sean or another student presented a correct solution immediately after. Sean was worried about violating student trust by asking a student to present work if he knew it was incorrect.

Jane - Interview. Like Sean, Jane said she felt that students were learning from worked example pitfalls, but that they had troubles learning from their own mistakes. She, too, felt that trust was an important aspect of the mathematical discourse in her class. Jane felt she would betray trust by asking students to present incorrect responses, "Yeah, I would feel really bad about seeing that a student had something wrong and saying, 'hey, go up and show the class.'" She also noted that pitfall discussions increased engagement, catching the "in-between kiddos," though she did not think any curriculum could engage students struggling in school in general. Unique to Jane was her focus on what individual students did or did not understand about

particular mathematical problems. For example, she was surprised when a student, discussing a worked example pitfall, made clear he did not understand what x represented, "he was saying 'how can you add a number, 15 to a number, if you don't know what that is?' And I was like 'huh? ... they're at a level where they should understand that."

Jane - Observation. Like Sean, Jane's class included some Level 1 and Level 2 sourcing of mathematical ideas. Students discussed in groups and then presented in class. She probed student thinking using prompts from the curriculum, and used wait time and asked "Anyone have anything to add?" to encourage engagement. She suggested people try, saying that mistakes are how people learn. These Level 2 attributes were mitigated by students' apparent (to Jane) lack of confidence in their ideas. It was sometimes hard to find volunteers and students often did not know how to respond to prompts. At these times, Jane would have students work more in groups while Jane coached an individual. The individual attention would continue until Jane and the student agreed the student was ready for presenting; meanwhile, other students became disengaged or struggled unproductively. Similar to Sean's classroom, if incorrect work from a fictional student in the materials or a student in the room was presented, someone quickly gave a correct solution without interrogating the sense-making behind the pitfall.

Rita – Interview. For Rita, pitfalls were something to be celebrated. Airing student mistakes and the shared value of pitfalls for learning lead to animated full class discussions. Students who thought they understood were pushed to learn more because confused students asked them questions. She said, "We clap if you do it wrong because you get it out of the way in classwork and homework and then you don't make that mistake on tests." The trust established within the class group appeared to support such celebration. According to Rita, when outsiders came to the class, students were less likely to share, "they didn't want to say anything wrong, so they were very careful and spoke a lot less than a typical regular lesson." Student engagement increased for many but not inclusively, for all students, in the experiences reported by Sean and Jane. Whereas for Rita, there was more equitable engagement, which she attributed, in part, to valuing pitfalls:

> I honestly believe that if you didn't know who my resource kids were and my other kids, you couldn't pick them out [...] In the past, you could have done that all year long. These kids now feel they have value in what they say, [...] so that has brought just a wonderful culture.

Rita – Observation. Rita was observed for only one typical class period, not the full two two-day MPP lessons required for the study. Considering this limitation, the data support Rita's descriptions of her class in which Level 3 elements were visible (even though Rita reported students were more reluctant to share due to the observer in the room). Rita put student thinking and examining misconceptions forefront: she shared with the class all ideas she heard discussed in group work and supported students' group and full class discussions. All students appeared actively engaged in small group discussions. In one instance she used a student misconception as an opportunity for further discussion. A couple of students shared the hypothesis that "when you multiply a number by a fraction, the product will be smaller." Rita had students discuss this in small groups during which time there was informal sharing between groups. In full class, a student shared "if you multiply by a fraction that is greater than 1 the product is going to be greater." One of the original students responded, "I changed my mind after looking at more examples." The observations provided evidence of the culture Rita described where students were willing to present novice understandings and all students engaged mathematically in discussions.

Discussion and Conclusions

Tension: Trust vs. Pitfalls. Sean, Jane, and Rita all mentioned that their students learn by making and thinking through mistakes. Sean and Jane were concerned with making student pitfalls public. Jane said, "When they go up to the front, they want to be right, because it is scary for them, whether or not you tell them a thousand times 'pitfalls are ok, it is part of learning.'" She thought students believed that only the correct answer is valuable and that she could not "undo" that in a few lessons. The observed full class discussions were centered around correct solutions or correcting a fictitious students' mistake, rather than centered on understanding student thinking around errors. This may have compounded student reluctance to share incorrect work. Sean and Jane seemed to struggle with the tension between valuing pitfalls for learning and upholding student trust and safety. In contrast, Rita reported students would not hesitate to share incorrect solutions or ask for help and the observation provided evidence of students' comfort sharing preliminary ideas without knowing about correctness. Making sense of pitfalls in class discussions and celebrating pitfalls seemed to be key differences between Rita's math-talk community and those of Sean and Jane and may be a key element of Level 2 sourcing of mathematical ideas (i.e., "using student errors as opportunities for learning"; Hufferd-Ackles et al., 2004, p. 89). For Rita, student trust was not about saving students from embarrassment around partially or wholly incorrect thinking, but rather about trusting that student ideas, whether pitfalls or not, were valuable for learning mathematics.

Source of Ideas: Teacher Goals and Student Actions. Sean wanted discussions focused solely on student ideas, with students questioning each other and the teacher keeping the conversation focused. On the surface this aligns with Level 3 sourcing of mathematical ideas. Similarly, Jane wanted student thinking and ideas to be foregrounded, but with the teacher taking a more prominent facilitation role, as described in Level 2. However, they both struggled when students did not volunteer to present or comment on each other's work. The students may have been expecting a Level 0 or 1 classroom while the teachers were aiming for Level 2 or 3. Authors of the Math-Talk framework described each attribute as being "developmental trajectories in teacher actions and students' actions [that] were derived from the data" (Hufferd-Ackles et al., 2004, p.87). Students and teachers likely must progress through each level in order, which may explain teachers' struggles to take their math-talk communities to the next level. Sean and Jane may have bypassed some of the transitional steps.

Pitfall-Focus May Create More Equitable Engagement. By focusing on student thinking, with particular attention to student reasoning around their own pitfalls, Rita believed she reached more equitable engagement and outcomes for students. Sean and Jane did not use student pitfalls as opportunities for learning, which may be related to their struggles to engage a wider variety of students. By focusing discussion on correct strategies, Sean and Jane were acting as arbitrators of what ideas were and were not valuable in the classroom, leading to less engagement. Jane grappled with this, "The idea of going up, they are going to want to work harder to make sure they are more accurate. But that's hard because you get kids who avoid doing the work because they don't want to [share work]." Spending more time validating, interrogating, and making sense of student thinking around pitfalls may lead to more and different students sharing.

Study Limitations. Our study conclusions were constrained by the completeness and inherent variability of our teacher self-report data: Sean did not complete part of the first interview due to time limitations, and though Rita participated in all interviews, we were only able to observe one typical day of one two-day MPP lesson. Finally, many factors that affect the nature of classroom

discourse, such as culture, language, race, and ethnicity (Moschkovich, 2007), were not investigated in the current study. As this was an exploratory study focused on discussion, not just pitfalls, teachers were not systematically asked about the complete set of themes that emerged relating to pitfalls, and we save this investigation for future research.

Educational Importance of the Research

Focus on Pitfalls. This exploratory study indicates that celebrating and focusing on reasoning behind student pitfalls may support quality student engagement and higher levels of math-talk. When pitfalls are discussed publicly and related to correct solutions, students may find sharing their own ideas and pitfalls to be valuable. However, teachers and students may be uncomfortable discussing student pitfalls and, thus, focus on correct solutions. Research should explore these dynamics in relation to the socio-cultural contexts they are embedded in.

Math-Talk Framework for Upper Grades and Equity. Sean and Jane's challenge to engage all students in discourse while maintaining trust is not explicitly addressed in the Math-Talk framework. Students in seventh grade may be more likely to worry about how they are perceived by peers, especially when making pitfalls, than the early grade students in the Hufferd-Ackles et al. (2004) work. Middle and high school teachers may benefit from tools that explicitly address how to build student trust in a math-talk community, with particular attention to maintaining trust while discussing what teachers perceive as sensitive topics such as students' pitfalls.

While the Math-Talk framework gave a lens for understanding the discourse in Sean, Jane, and Rita's classroom as a whole, it does not address individual student engagement. Based on these findings, class level engagement builds on individual engagement and is an important dimension of classroom discourse. For example, not all students engaged in quality math-talk in Sean and Jane's classrooms. The research community, and likely teachers, could benefit from a mathematics classroom discourse framework that could pick up on differences in student engagement within a classroom.

Supporting Math-Talk Community Development. This study points to some areas of focus for teacher educators and curriculum developers in helping support teachers in developing a math-talk learning community. First, math-talk communities develop over time, likely going through each of the developmental Math-Talk framework levels. Perhaps teachers should not be encouraged, as in this study, to engage students in Math-Talk framework Levels 2 and 3, without intentionally moving students first through Levels 0 and 1.

Second, teacher educators and curriculum developers can help teachers and students become more skillful in focusing discussion on the reasoning behind pitfalls. In particular, students may learn the content more deeply through interrogating pitfalls as the teacher supports a community norm in which all students' ideas are valuable for learning.

Acknowledgements

This material is based upon work supported by the National Science Foundation under Grant No. DRL-1314416. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation. Special thanks to Aleata Hubbard, Katie Salguero, Yvonne Kao, Shandy Hauk, and the case study teachers for their contributions to this report.

References

Barbieri, C., & Booth, J. (2016). Support for struggling students in algebra: Contributions of incorrect worked examples. *Learning and Individual Differences*, *48*, 36–44.

Booth, J., Lange, K., Koedinger, K., & Newton, K. (2013). Using example problems to improve student learning in algebra: Differentiating between correct and incorrect examples. *Learning and Instruction*, *25*, 24–34.

Hufferd-Ackles, K., Fuson, K., & Sherin, M. (2004). Describing levels and components of a math-talk learning community. *Journal for Research in Mathematics Education*, *35*, 81–116.

Michaels, S., O'Connor, M., & Resnick, L. (2008). Deliberative discourse idealized and realized: Accountable talk in the classroom and in civic life. *Studies in Philosophy and Education*, *27*(4), 283–97.

Moorjani, A., & Kao, Y. (2019). *Math Pathways & Pitfalls Algebra Readiness: Lessons and teaching guide,Grades 7-8 (beta version)*. WestEd. Retrieved from https://www.wested.org/resources/mpp-beta

Moschkovich, J. (2007). Examining mathematical discourse practices. *For the Learning of Mathematics*, *27*(1), 24–30.

Moschkovich, J. (2013). Principles and guidelines for equitable mathematics teaching practices and materials for English language learners. *Journal of Urban Mathematics Education*, *6*, 45–57.

Murata, A., Siker, J., Kang, B., Baldinger, E., Kim, H.-J., Scott, M., & Lanouette, K. (2017). Math talk and student strategy trajectories: The case of two first grade classrooms. *Cognition and Instruction*, *35*(4), 290–316. doi: 10.1080/07370008.2017.1362408

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. National Council of Teachers of Mathematics.

Patton, M. (2015). *Qualitative research & evaluation methods: Integrating theory and practice*. Sage.

Walshaw, M., & Anthony, G. (2008). The teacher's role in classroom discourse: A review of recent research into mathematics classrooms. *Review of Educational Research*, *78*(3), 516–551.

Willey, C., Gatza, A., & Flessner, C. (2017). Mathematics discourse communities: Language ideologies and urban mathematics teaching with Latinas/os. *Journal of Cases in Educational Leadership*, *20*(1), 34–48.