Journal of the California Mathematics Project
Volume **7**, 2015

Deepening Student Understanding of Area and Volume by Focusing on Units and Arrays

Allison Dorko and Natasha Speer

Introduction

When we teach elementary school students about area, surface area, and volume, we are teaching ideas that help them measure the physical world. In biology and chemistry, the surface area to volume ratio sets a limit to the size of cells, and is also a factor in the rate at which chemical reactions occur. Volume is important in measuring density, and in calculus, understanding area and volume is critical for understanding the construction of Riemann sums that are used to model accumulation of various quantities. In addition to these applied uses in science, technology, engineering and mathematics (STEM) fields, by teaching students these ideas, we are also giving them tools with which to understand other mathematics. For example, beyond serving as a measure of space, area can be used as a model for multiplication of real numbers and multiplication of expressions with variables (see Figure 1).

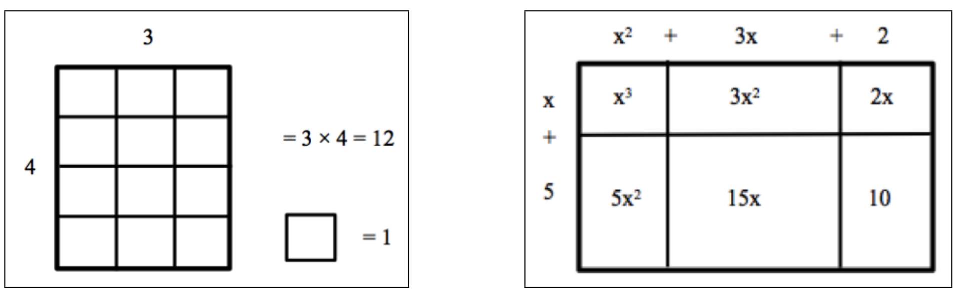


Figure 1. Examples of area models as they are sometimes used to illustrate multiplication of numbers (left) or multiplication of algebraic expressions (right).

The Common Core State Standards recommend that mathematics in early childhood focus on number, geometry, spatial relations, and measurement (Council of Chief State School Officers [CCSSO], 2010). Included in this is understanding units of measure, which are notoriously tricky for students of all ages. Given that units of measurement can serve as clues about how to interpret and combine quantities, helping students understand and make use of units gives them a tool for problem-solving and mathematical modeling.

Students of all ages can have difficulty understanding these important ideas and even those who carry out computations accurately may not have a rich understanding of the concepts of area and volume (Battista & Clements, 1998; Simon & Blume, 1994). We know that many students do develop good understanding of these ideas by the time they reach college. However, those who do not may face particular challenges when the instruction they receive makes substantial use of units, area, and volume. To better understand these challenges, we have been investigating college students' understanding of area and volume and associated units of measurement. This work provides mathematics instructors with information about student understanding and can be used to identify the difficulties that persist for some students. In particular, we have used it to inform the design of activities for secondary school students that may help strengthen student understanding of area, volume, and their measurement.

Research Findings

Area and volume computations are based on the idea of iterating unit squares (or cubes, as appropriate[1]) into rows and columns (or rows, columns, and layers) such that there are no gaps or overlaps. While adults may perceive these rows, columns, and layers as an organized structure, some learners may see an array of unit-sized pieces as randomly arranged objects (Battista & Clements, 1998). For these students, the number of unit-sized pieces do not represent a spatial measure. Difficulties with perceiving the structure of an array and with spatial visualization may lead to students not counting the innermost cubes of an array (e.g., counting 26 cubes in a 3×3 array like a Rubik's cube because the centermost cube is not visible) and double-counting the edge or corner cubes. In other words, students may see a three-dimensional array in terms of its faces and neglect the "middle" (Battista & Clements, 1998). Not perceiving area and volume as arrays may explain why many students, including preservice teachers, cannot explain why a formula like $A = lw$ generates a measure of area or why $V = lwh$ gives a measure of volume (Simon & Blume, 1994). Relatedly, identifying the shape of the cross-section of a solid can be difficult for middle school and high school students (Davis, 1973), which prevents them from thinking about the volume of a rectangular solid as the area of the base times the height. The good news is that students who do understand arrays tend to be successful with area and volume computations (Curry & Outhred, 2005; Dorko & Speer, 2013a, 2013b).

Units of measure are also difficult for students of all ages. Elementary school students tend to misappropriate units of length for area, volume, and angle (Lehrer, 2003). Researchers have replicated this finding with different tasks and concluded that some students do not see the need for a unit of cover in area measurements. Instead, students with this view will add lengths together to get the "area." For example, some children measure the area of a square by measuring one side of the square, moving the ruler a short distance parallel to the side of the square, and measuring the lengths of the two sides perpendicular to the ruler with each successive horizontal movement (Lehrer, 2003). Misappropriating length units for other spatial measures seems indicative of trouble with dimensionality (i.e., length is one-dimensional but area is two-dimensional). If a learner does not perceive area as composed of unit squares and volume as composed of unit cubes, then expressing areas and volumes in square and cubic units can seem arbitrary.

Though one would hope, or even assume, that undergraduates understand area, volume, and units, our research suggests otherwise. We began studying student learning about volumes of solids of revolution and optimization in calculus. It turned out that in many cases, students' sparse understanding of area, volume, and surface area prevented them from completing the mathematical modeling needed for using spatial accumulation ideas from calculus. To look more deeply into how students thought about area, volume, and units in non-calculus contexts, we gave basic area and volume computational tasks (areas of a rectangle and circle; volumes of a rectangular prism, a cylinder, and a right triangular prism) to 198 calculus students. Seventy-three percent of students gave an incorrect unit for at least one task (Dorko & Speer, 2014, in press). We found that students struggled with computing volumes. Some students computed surface area instead of volume, which is reminiscent of elementary school students' tendency to think about only the faces of an object when they enumerate a volume array. Finally, some students used formulae that contained both surface area and volume elements (which we term "amalgam" formulae). Table 1 (next page) categorizes some student responses to a problem that directed them to find the volume of a cylinder of radius r and height h. Note that some formulae fall into two rows. For example, some students said that $2\pi r^2$ found the area of the base, and multiplying by the height gave them volume (putting this in the second row) while others described the two as accounting for the "two bases" (a surface area idea).

[1]A "unit square" is a covering shape (interior and edges) in the plane that is bounded by four sides of equal length with four equal interior angles and a "unit cube" is the solid (interior, faces, and edges) bounded by six unit square faces with three meeting at each vertex.

Table 1. Student-Produced Formulae for the Volume of a Cylinder of Radius r and Height $h^\dagger$

Correct	**Incorrect**			
Volume Formula	*No surface area element*	*Amalgam of surface area and volume elements*	*Surface area*	*Other*
$\pi r^2 h$	$2\pi r^2 h$	$2\pi r^2 h$	$2\pi r^2 h + 2\pi r h$	$d + h$
	$\frac{1}{3}\pi r^2 h$	$2\pi r h$	$2\pi r^2 h + \pi d h$	
	$\frac{1}{2}\pi r^2 h$	$2\pi r + \pi r h$		
	$\frac{4}{3}\pi r^2 h$	$\pi r^2 + 2\pi d$		
	$\pi r h$	$2\pi r^2 + 2rh$		
	$\frac{1}{2}\pi r h$			
	hdr			

† From Dorko and Speer, 2013b (p. 54).

Sometimes teachers and professors assume that students simply remembered a formula incorrectly. Our results indicate otherwise. The students we interviewed explained the formula they offered. For example, one student said, for $2\pi r + \pi r h$, that $2\pi r$ gave the measure of the areas of the bases and $\pi r h$ accounted for the space between them. We suggest that teachers view an incorrect formula as a likely symptom of not understanding area, volume, and/or surface area. For instance, suppose a student used the formula $2\pi r^2 h$ to find the volume of a cylinder. In responding to the student's work, one might be inclined to cross out the two, assuming that the student "forgot" that the area of a circle is simply πr^2 and not $2\pi r^2$. If the student had written $2\pi r^2 h$ knowing that the area of a circle was πr^2 but thinking the volume calculation needed to "include both bases" (as some of our students explained for the $2\pi r^2 h$ formula), a teacher crossing out the two or writing the correct formula does nothing to help the student better understand volume. Instead, we suggest using students' formulas as clues to how they are thinking about volume.

In the next section, we provide other ideas for instruction. These ideas are based around connecting units of measure to other ideas in mathematics, such as rules of exponents, and on tasks that build students' understanding of arrays so that they can come to see the volumes of rectangular solids as the area of the base times the height.

Ideas for Instruction

Table 2 lists some of the findings from our research, the implications for instruction, and references to tasks that might be used to strengthen student understanding. The tasks are in the print-ready Worksheets that start on page 11. See page 14 for accompanying teacher notes.

Conclusion

In summary, we have seen that some of the issues that elementary, middle, and high school students experience regarding area, volume, and units persist into the undergraduate years. Of course, this is not true for all students. On the whole, undergraduates in our work were more successful with area and volume computations, and explaining area and volume formulae, than what research has reported for younger students. The undergraduates who were successful with these computations and who could unpack and explain a formula generally did so because they conceived of areas and volumes as arrays. We suspect that thinking about arrays makes it easier for students to understand why area is in units squared and volume is in units cubed. Early facility with units provides students with a tool to help them understand and carry out calculations in chemistry, physics, differential equations, and other areas of mathematics. In short, mathematical measurement is a way to see and appreciate the world.

Table 2. Research, Instruction, and Tasks for Classroom Use

Research Finding	*Instructional Implication*	*Tasks*
Students struggle with the units for area and volume computations.	Connecting squared/cubed units to laws of exponents provides students with an example of laws of exponents in the real world. For instance, show that the volume of a cube with side length 5 is $(5\text{ cm}^1 \times 5\text{ cm}^1 \times 5\text{cm}^1) = 5^3\text{cm}^3$. Emphasize that a linear unit has an exponent of 1, just like 7 can be written as 7^1 and x can be written as x^1, and hence the "when you multiply, you add exponents" rule applies to units as well as numbers and variables.	A, B
Students struggle with understanding arrays and that the volume of a rectangular solid can be thought of as $V = Bh$ where B is the area of the base.	Most volume computations that elementary and secondary school students are asked to complete can be carried out with an "area of base times height" approach. However, students often encounter specialized versions of this for shapes such as cubes, rectangular prisms, and cylinders (s^3, lwh, and $\pi r^2 h$ respectively). Emphasize that these formulae are special cases of the more general approaches that also better illustrate the idea of layers or arrays. For instance, writing the volume of a cube as $s^2 \cdot s$ may make it clearer to students that s^2 is the area of the base and s is the height.	C, D, E, F

References

Battista, M. T., & Clements, D. H. (1998). Students' understandings of three-dimensional cube arrays: Findings from a research and curriculum development project. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (p. 227-248). Mahwah, NJ: Erlbaum.

Common Core State Standards for Mathematics. (2010). Washington, DC: Council of Chief State School Officers [CCSSO].

Curry, M., & Outhred, L. (2005). Conceptual understanding of spatial measurement. In P. Clarkson (Ed.), *Building connections: Theory, research and practice (Proceedings of the 27th MERGA conference).* Sydney, Australia.

Davis, E. (1973). A study of the ability of selected school pupils to perceive the plane sections of selected solid figures. *Journal for Research in Mathematics Education*, *4*, 132-140.

Dorko, A., & Speer, N. (2013a). Calculus students' understanding of volume. In S. Brown, G. Karakok, K. H. Roh, & M. Oehrtman (Eds.), *Proceedings of the 16th annual conference on research in undergraduate mathematics education* (Vol. 1, p. 190-204). Denver, Colorado. Retrieved from `http://sigmaa.maa.org/rume/RUME16Volume1.pdf`

Dorko, A., & Speer, N. (2013b). Calculus students' understanding of volume. *Investigations in Mathematics Learning*, *6*, 48-68.

Dorko, A., & Speer, N. (2014). Calculus students' understanding of units. In T. F. Connolly, G. Karakok, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 17th annual conference on research in undergraduate mathematics education* (p. 584-592). Denver, Colorado. Retrieved from `http://sigmaa.maa.org/rume/RUME17.pdf`

Dorko, A., & Speer, N. (in press). Calculus students' understanding of area and volume units. *Investigations in Mathematics Learning*.

Lehrer, R. (2003). Developing understanding of measurement. In J. Kilpatrick, W. G. Martin, & D. E. Schifter (Eds.), *A research companion to principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Simon, M. A., & Blume, G. W. (1994). Building and understanding multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education,*, *25*, 472-494.

Worksheet Pages

A

Rewrite the following using exponents:

$5 \cdot 5 \cdot 5 \cdot 5 =$ ________

$a \cdot a =$ ________

$\text{cm} \cdot \text{cm} =$ ________

Expand the following as products so they are written using only terms whose exponents are 1:

$7^4 =$ ________

$x^7 =$ ________

$\text{ft}^3 =$ ________

Would it make sense for the area of this piece of paper to be 93.5 inches? Why or why not?

B

For each measurement given, pose a problem whose *answer* is the spatial measure listed. (You can think of this like the game *Jeopardy*, where you are given the answer and you need to design the question).

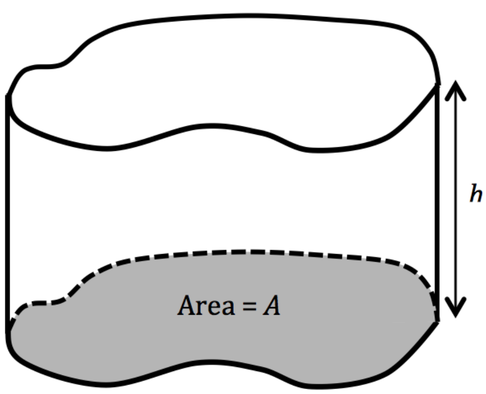
8 cm^3

4π cm

$9\pi \text{ cm}^2$

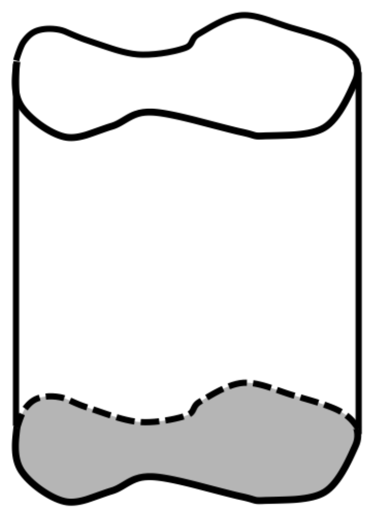
$9\pi \text{ cm}^3$

C



If the area of the base of this prism is 44.1 cm^2 and the height of the prism is 7.3 cm, what is the volume?

D



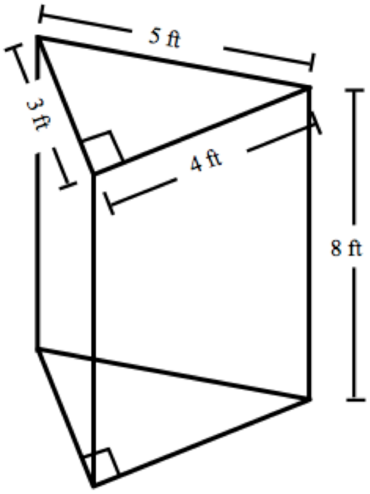
If the volume of this prism is 56 in^3 and its height is 7 in, then which of the following is the area of the base?

(a) 8 in (b) 8 in^2 (c) 392 in^2

(d) 392 in^3 (e) 8π (f) $8\pi \text{ in}^2$

E

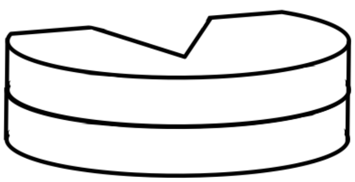
Find the volume of the triangular prism pictured below.



F

Consider the shape below. The area of its base is 6 in^2 and its height is 1 in. What is its volume?

Now suppose we have many of these shapes, and we stack them to form bigger shapes.



If we stack two of them together, what is the area of the base?

What is the height?

What is the volume?

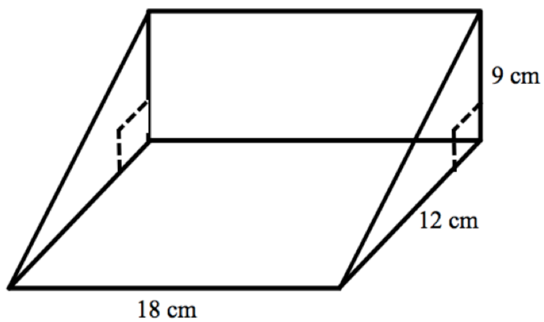
Suppose we continue to stack these shapes. Complete the table below.

Number of Layers	Area of Base	Height	Volume(Area of Base $\times$ Height)
1			
2			
3			
4			
$\vdots$			
n			

What is the volume of the shape formed by stacking n layers?

G

While the word "base" in English usually means the "bottom," in mathematics the base of a shape may not actually be the face that looks like it is resting on the "ground." For instance, the prism below is considered a *right triangular prism* with a base that is a triangle.

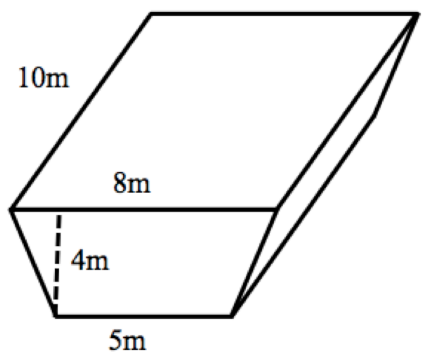


(a) What is the area of the base of this prism?

(b) What is the volume of this prism?

H

Consider this prism:



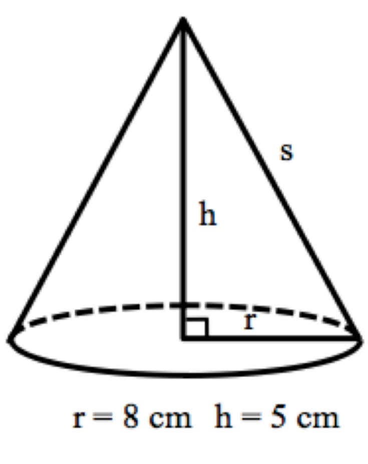
(a) Shade the prism's base.
(b) What is the name of this prism?

(c) Find the area of the base.

(d) Find the volume of the prism.

I

A friend looks at the following shape and tells you that the area of the base is $25\pi\ \text{cm}^2$ and that the volume is $25\pi\ \text{cm}^2 \cdot 8\ \text{cm} = 200\pi\ \text{cm}^3$.



(a) Did your friend find the correct volume? Why or why not?

(b) Does your friend's computation have the correct units for volume? Why or why not?

Teacher Notes on Worksheet Tasks

A

For the tasks:

cm · cm = ________
ft^3 = ________

some students may answer:

cm · cm = c^2m^2
$\text{ft}^3 = \text{f}^3\text{t}^3$ or f·t·t·t

This creates an opportunity to discuss various mathematical conventions that involve abbreviations for units.

B

This sequence of tasks:

$9\pi\ \text{cm}^2$
$9\pi\ \text{cm}^3$

provides opportunities for students to make connections between the value of the exponent on the unit and the physical properties of the shape. For instance, you might help students notice that appropriate problems to pose for these two answers could be the area of a circle with radius 3 cm, and the volume of a cylinder with radius 3 cm and height 1 cm. It may be helpful to point out that 9π cm can also be written as $9\pi\ \text{cm}^1$.

C

Shapes such as the one in this task, where students cannot readily compute the area of the base, reinforce the use of the forumula:
Volume = (area of the base) × (height)
also written: $V = Bh$.

D

As an extension for item **D**, you can ask students to describe the kind of object that each of the wrong answer choices could refer to.

E

Students may try to multiply all the given numbers in this problem.
Suggest that students include units throughout computations. This practice, common in chemistry and other sciences, would look like this:
$3\text{ft} \cdot 4\text{ft} \cdot 8\text{ft} = 96\text{ft}^3$

F

Students may have trouble recognizing that they have been given the area of the base of the layer, and they may try to compute the area of the base. The goal of this task is that students gain practice with the Volume = (area of base) × (height) strategy, and generalize this for a shape with height n units.

G & H

Students may need help naming shapes. Manipulatives may be helpful here, as they allow students to move prisms into a more recognizable orientation.

I

Some students may think $200\pi\ \text{cm}^3$ is correct because it is the area of the base times the height. The goal of this question is to help students attend to the limitations of the $V = Bh$ strategy for finding volume.

About the Authors

Allison Dorko ⟨dorkoa@onid.oregonstate.edu⟩ is a doctoral candidate in mathematics education at Oregon State University. Her research focuses on how students learn calculus, with particular attention to generalization and the transition from single to multivariable calculus. During the summer, she works at a summer camp, where she sneaks mathematics into the campers' day without their realizing it.

Natasha Speer ⟨natasha.speer@maine.edu⟩ is an associate professor of mathematics education at the University of Maine. Her research focuses on the teaching and learning of undergraduate mathematics, knowledge used in teaching, and teacher professional development.