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Pathways and Pitfalls on the Road to Equity in Mathematics Classrooms

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Abstract. The article describes the foundational principles and central design features of the *Math Pathways & Pitfalls* (MPP) intervention curriculum for elementary grades and illustrates challenges in creating MPP-like middle school algebra readiness materials. We describe key design features and discuss how they support language, discourse, and equity development. The article offers vignettes created from recent middle school classroom observations and teacher interviews to illustrate the particular instructional pathways and pitfalls emerging in the pilot of MPP-like algebra readiness student and teacher materials.

Background

Math Pathways & Pitfalls (MPP) is a K-8 intervention and enrichment curriculum. It was developed and field-tested with grants from the National Science Foundation, U.S. Department of Education, and Stuart Foundation. The materials have broad appeal, addressing some of the toughest math concepts for learners. Grades K to 3 materials focus on developing whole number concepts and operations, while grades 4 to 8 focus on rational number concepts and operations (Barnett-Clarke, Ramírez, & Coggins, 2010). In developing *MPP*, the authors drew on a combined 35 years of teaching experience and combed the research literature on student conceptions about mathematical ideas. Pitfalls reported in that research are the basis for each of the ten lessons in a grade-level unit.

Lesson Format. Each lesson has a set of common components, but presented in ways that are not particularly traditional. Indeed, the approach can lead to questions and push back from teachers or professional developers, until they see how discourse communities (of students and teachers) emerge from the use of the materials. A lesson begins with vocabulary and individual student thinking on a Starter Problem. Then, learners consider and discuss problem-solving processes of two student characters, one OK and the other a Pitfall (Figure 1). In reflecting on and discussing why incorrect solutions are not correct, learners explore their own problem solving and that of the characters. Students have practice in a mindset that views errors as learning opportunities. After an extended consideration of the Starter Problem,

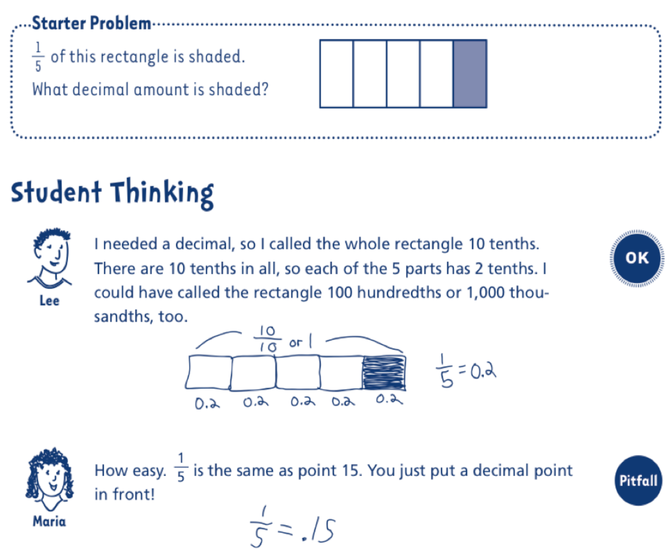


Figure 1. Sample Starter Problem for *Opener* and *Discussion*.

students work alone and with others to review and practice the central concept in a lesson by working on and discussing additional problems. *Math Pathways & Pitfalls* lessons are designed to take place across three days, about $2\frac{1}{2}$ hours total (see Figure 2).

Core Lesson: Day 1	Core Lesson: Day 2	Mini-Lesson: 2 to 3 days later
Opener: Students review Discussion Builders, new math words, and do Starter Problem on their own. **Discussion**: Students talk about OK and Pitfall, list Things to Remember.	**Review**: Students review the previous day's lesson and Things to Remember. **Practice**: Students work with a partner to solve 3 problems and explain their methods, then work 3 problems on their own.	**Assess**: Students do multiple-choice problems and justify their solutions in small groups. **Reinforce**: Students practice individually and in pairs, writing a complete and correct solution and explanation.

Figure 2. *Math Pathways & Pitfalls* lesson structure.

The instructional package consists of student pages, teacher guide, videos modeling productive discussions for students (teachers show these in class), CDs with printable versions of formative and summative assessments, and a 3' x 2' *Discussion Builders* poster (see Figure 3 for an excerpt). The materials are designed to foster a robust understanding of mathematics concepts that does not cave in to "pitfalls" as well as to enhance students' ability to articulate mathematical ideas, justify their reasoning, and develop their metacognitive abilities.

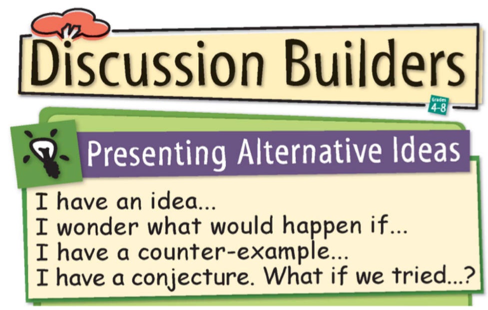


Figure 3. Example from *Discussion Builders* poster.

Teacher Resources. The same design features that shaped student materials have informed face-to-face and *Principles to Practice* web-based teacher professional development materials for *MPP* (Ramírez et al., 2016). The materials are simple. There are few technology requirements, beyond a web browser, for the teacher online professional development. None of the student or teacher pages are in color and manipulatives are sometimes suggested, but not required. Most of the "mathematical magic" happens in the discussion of a math problem and potential correct and incorrect solutions to that problem. The teacher resources highlight building skill with teaching practices that are effective for orchestration of classroom discussion, in small groups and whole-class.

Effectiveness of *MPP* Materials. Two large randomized trials have evaluated the impact of *Math Pathways & Pitfalls* at various elementary grade levels and exploratory studies have provided insight into how the lesson structures lead to improved math achievement and language development for learners. Two experimental studies among 4,000 economically and ethnically diverse students in grades 2, 4, 5, and 6 found significant positive effects for students – including Latino/a and English language learners – when teachers used seven *MPP* lessons across one year (Heller, et al., 2006, 2009, 2010; Khisty & Radosavljevic, 2010). Study effect sizes were .43 to .66 standard deviations, a gain representing 50 percent or more improvement over the annual gain otherwise expected in mathematics for students in those grades (Lipsey et al., 2012, p. 28, T5). It is clear from the studies that the materials have replicable success in demographic groups traditionally underserved by conventional materials.

Building on Success. Since they were developed, the original four *MPP* books have been used by hundreds of teachers and thousands of students in elementary school, multiple-subjects classrooms. The theoretical underpinnings have been foundational in other projects, including the NSF-funded *Using Math*

Pathways & Pitfalls to Promote Algebra Readiness effort now underway (DRL-1314416). In that project, a fifth book is in the works. As of this writing, the new algebra readiness materials are being pilot-tested. Revised *MPP-AR* materials and associated teacher professional development will be field-tested in the 2018-19 school year. Here, we report on the controversial design features of *MPP* and the challenges inherent in creating materials for middle school students and their teachers that use the *MPP* approach.

Extending *MPP* to Algebra Readiness in Grade 7 Contexts

Classroom equity requires practices by teachers and students that support every person to participate in mathematically rigorous and productive conversations. The burden on developers is to include opportunities to learn in the materials that are experienced by students *as* opportunities. The goal is that students feel invited to enter into a discussion where their many linguistic, cultural, and educational backgrounds are resources for contributions. And, in the current *MPP Algebra Readiness* (*MPP-AR*) project, these opportunities must be tuned to the realities of middle school single subject classrooms.

***MPP* Principles.** *MPP* and *MPP-AR* are based on five foundational principles:

Building Mathematical Discussions: Using academic language to reason about, explain, and justify mathematical ideas builds understanding and the capacity to make mathematical arguments.

Making Sense: Making sense of the mathematical meaning of words, symbols, and diagrams in contextualized and decontextualized problems is fundamental to strategizing, implementing, and evaluating solutions.

Confronting Pitfalls: Contrasting mathematical reasoning with and without pitfalls builds conceptual understanding and prompts students to self-monitor and self-correct.

Visualizing and Connecting: Discussing relationships among mathematical ideas using visual, verbal, and symbolic representations builds robust conceptual understanding.

Capturing Key Ideas: Creating a strategic public record of key mathematical ideas as they are being discussed helps students understand, summarize and remember those ideas.

The goal is that students better understand, reason about, and articulate key mathematical ideas. While the principles were not called out separately in the original *MPP* books, they are in *MPP-AR* materials.

Students arrive in middle school with distinct pools of knowledge from half a lifetime of varied experiences in mathematics teaching and learning. These pools may or may not be connected for students. Thus, to serve students in the single subject classroom context, teachers need mathematical knowledge for teaching that includes nuanced understanding of mathematics content as well as deep knowledge about how students think. Shaped by the existing principle-driven *MPP* teacher materials, the *MPP-AR* book and online professional development provide resources for teachers to organize what they notice in the classroom and connect it to what research has demonstrated about ways students think. At the request of teachers, in each lesson the *MPP-AR* book includes accessible summaries of relevant research on aspects of student thinking.

The following sections describe the central design features of the *MPP* approach. These features are how the MPP principles are realized in the design: discussion builders, contrasting cases, multiple representations of student thinking, examples of divergent approaches, intellectual risk-taking, and scaffolds for classroom discussions. These are present in the original *MPP* curricula and have driven the design of *MPP-AR* development. For each feature, we describe it, give the controversy about it, tell the backstory for the approach, and close with some instructional pathways that support language, discourse, and equity. Examples come from both the original *MPP* and the new *MPP-AR* materials. To illustrate the contrasts across *MPP-AR* and *MPP*, we also have included vignettes created from recent *MPP-AR* classroom observations and teacher interviews (student and teacher names are pseudonyms).

Design Foundation: *Discussion Builders*

Every *Math Pathways & Pitfalls* lesson for students and professional development endeavor includes the use of *Discussion Builders* (see Figure 3, above). The *Discussion Builders* are sentence stems based on work from the Strategic Literacy Initiative. These stems encourage students to engage in discourse practices that include listening, speaking, argumentation, reasoning, and justification (Braunger, Donohue, Evans, & Galguera, 2004; Hinchman & Appleman, 2017). The poster hangs in the classroom all the time. It is a shared referent for students and teacher. Classroom observation studies documented the power of classroom use of *Discussion Builders* to support students to think, write, draw, and talk about math ideas (Heller, et al., 2006). In fact, the posters proved useful for non-*MPP* lessons (Heller et. al, 2009).

Design Feature 1: Contrasting Cases - OK and Pitfall

Research Foundation. Contrasting examples facilitates transfer to new problems and helps memory retrieval (Gentner, Loewenstein, & Thompson, 2004). Discussion is deepened by contrasting two examples of student thinking. Consider the Starter Problem in Figure 1, with one answer that is correct (Lee) and one based on a pitfall (Maria). Students discuss each example of student thinking, carefully unpacking the meaning of each example students' written statements, symbols, and drawings. Understanding is reinforced by talking about Lee's correct solution, as well as by determining what is awry in Maria's incorrect solution.

Controversy for OK and Pitfall/Oops. Why give students the answer? Why talk about a correct solution first, and the proposed incorrect solution second? Why talk about mistakes? Won't students make them more often if we expose them?

What are pitfalls? Pitfalls are stubborn incorrect or incomplete conceptions about mathematical ideas. Many pitfalls are highly predictable and salient, but rarely talked about or confronted. We have found that confronting pitfalls:
(a) produces cognitive dissonance,
(b) stimulates inquiry, and
(c) motivates a metacognitive stance towards learning (e.g., a growth mindset). As an example, consider the starter problem in Figure 4. The majority of first grade students will have one of the two typical pitfalll responses on this problem: 687 or 138 (Moss & Case, 1999). We used the latter as the Grade 1 pitfall. The word "Oops" is used in K through 3 books with "Pitfall" introduced and used for Grade 4 and above.

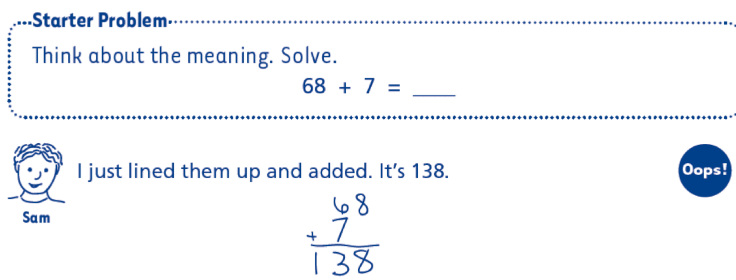


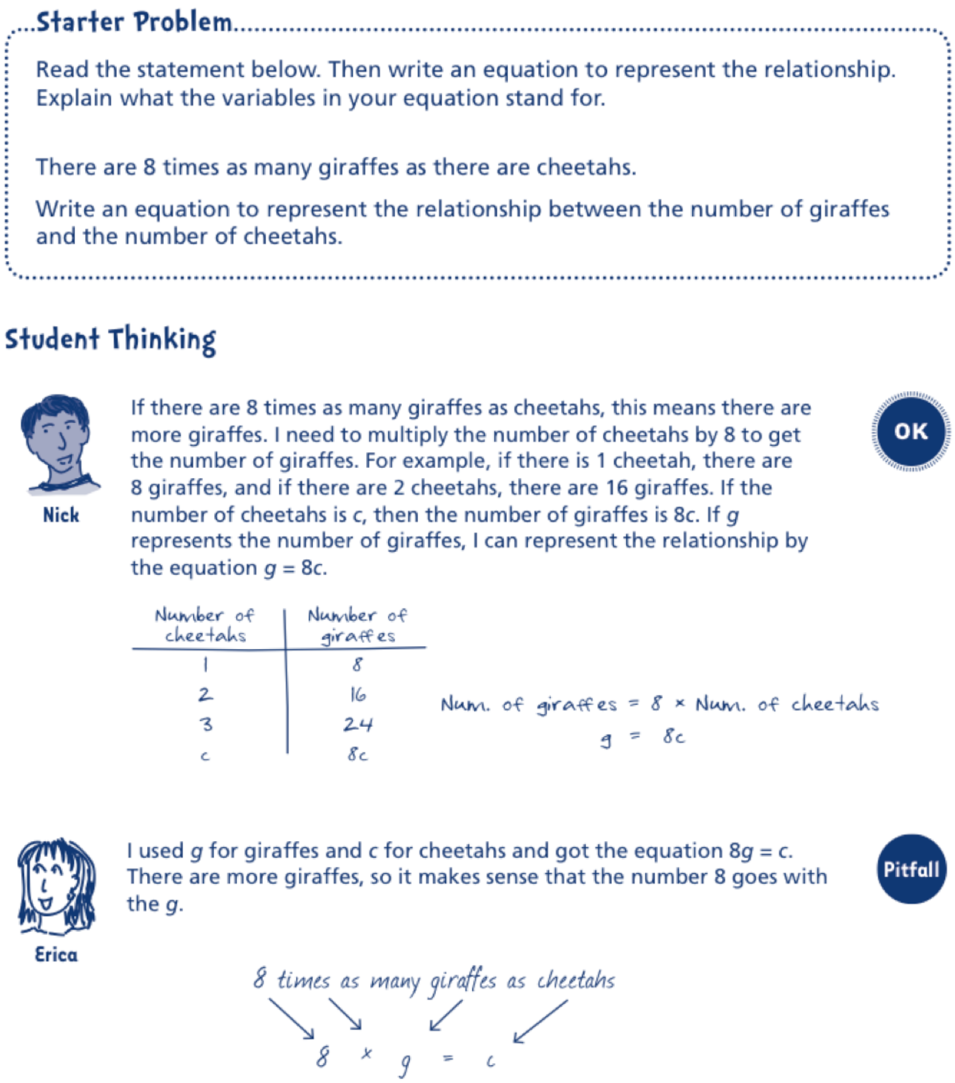
Figure 4. Starter Problem and related Pitfall (Oops!) solution.

Backstory. There are many excellent curricula out there that provide problems for students to solve. Our goal here was to make the task about understanding solution processes, having students explain the reasoning behind them, and justify why one is correct and the other has some pitfalls. The primary focus is on the *meaning* of the problem and *understanding the mathematical structures* underlying a solution, rather than on producing an answer. *MPP* lessons are not just about preventing pitfalls. They are about developing so robust an understanding of a key concept that it does not cave in to pitfalls. Students are expected to bring meaning to contextual, as well as purely symbolic problems.

Students can learn as much about the underlying concepts from the example with a pitfall as from a correct solution. Conversation might focus on **number sense**: Is Sam's solution too high or too low? How do you know? Or on **place value**: What does the 7 mean in Sam's solution?

Example from *MPP* Algebra Readiness. At higher grade levels, much more language is used in justification. Examples of how students explain their thinking illustrate the usefulness of attempting an explanation. Thorough justification is a way to uncover where a solution derails.

Consider the starter problem and example solutions in Figure 5. The problem was selected to foreground the well-documented pitfall of direct, sequential, translation of natural language to mathematical representation.



Starter Problem

Read the statement below. Then write an equation to represent the relationship. Explain what the variables in your equation stand for.

There are 8 times as many giraffes as there are cheetahs.

Write an equation to represent the relationship between the number of giraffes and the number of cheetahs.

Student Thinking

Nick

OK

If there are 8 times as many giraffes as cheetahs, this means there are more giraffes. I need to multiply the number of cheetahs by 8 to get the number of giraffes. For example, if there is 1 cheetah, there are 8 giraffes, and if there are 2 cheetahs, there are 16 giraffes. If the number of cheetahs is *c*, then the number of giraffes is 8*c*. If *g* represents the number of giraffes, I can represent the relationship by the equation *g* = 8*c*.

Number of cheetahs	Number of giraffes
1	8
2	16
3	24
c	8c

Num. of giraffes = 8 × Num. of cheetahs

g = 8c

Erica

Pitfall

I used *g* for giraffes and *c* for cheetahs and got the equation 8*g* = *c*. There are more giraffes, so it makes sense that the number 8 goes with the *g*.

Figure 5. Example algebra readiness Starter Problem with OK and Pitfall solutions.

Algebra Readiness Vignette. Part of development for the *MPP-AR* materials has been observing in-the-wild classroom interactions that result from such pilot problems. The vignette below is from a Grade 7 class discussing the cheetahs and giraffes problem from Figure 5. The *MPP*-recommended process for the starter problem is to have students work on individual solutions, pair up and share with one other person, then turn the page to look at and talk about the OK and Pitfall solutions in pairs. Subsequently, the teacher facilitates a whole class discussion. Instead of using this approach, the teacher in the vignette watched students work, found a student who did the problem incorrectly, and asked that student to come to the board to start conversation. Though the intent was to generate conversation about a pitfall, the teacher did not first have students make sense of a correct approach. The result was that the Pitfall and OK solutions made it into the air in the room, but the Pitfall came first and some confusion and counter-productive effort used class time that might have been better spent using the *MPP* discourse format.

Ms. Lee's Grade 7 Class

Students have made their first attempts, individually, to solve the starter problem. As the teacher, Ms. Lee, circulates around the room, she speaks to a student about presenting work to the class. The student, Alicia agrees to do so. Alicia's work shows:

g=giraffe c=cheetah $c = 8 \times g$

Ms. Lee says to the whole class, "Use your Discussion Builders to ask Alicia a question or share your thoughts." Lana says, "I am confused on how do you get the answer." Alicia, standing at the front of the room, responds, "So if you have 2 giraffes then there are 16 cheetahs." A third student, Jaime, shares, "I disagree with Alicia's answer because there are 8 times as many giraffes as there are cheetahs."

Students start making noise around the disagreement. The teacher changes the context of the problem in an attempt to help students make sense of the starter problem. "If you were a guy would you want to go to a dance that had 8 times as many guys as girls or 8 times as many girls as guys?"

The class tries to makes sense of the new context but students still answer that both situations seem like they could be the answer. The class goes back to discussing the starter problem context. Another student, Pete, states his disagreement with Alicia's solution and goes to the front to present his approach to the class using the document camera. "I disagree because it says 8 times as many giraffes than cheetahs. I started off with using the number 8. So if I have 8 cheetahs then I have 64 giraffes because of the statement in the problem." Pete's work:

Cheetahs x 8 = giraffes

c x 8 = g so c8 = g so 8c = g

Nira raises her hand and says, "I agree with Pete because it seems clear to me." Some students are saying "What?" or shaking their heads in confusion. A fifth student asks Pete, "How did you get that answer?" and Pete responds with the same example: if there are 8 cheetahs then there will be 64 giraffes.

All of the sudden Alicia (who is still standing near the front of the room) says, "OH! I get it now! I know that if I have one cheetah then I am going to have 8 giraffes."

Ms. Lee asks Alicia, "Is there a pictorial way for you to represent that?" Alicia goes up to the board and writes 8:1. She looks to the teacher (who is mid-room) from the front of the class and states, "I get it, but I am not sure how to write it."

Ms. Lee talks and Alicia writes as Ms. Lee creates a two-column table representing the relationship Pete showed the class. Alicia and Pete sit down and Ms. Lee has students turn the page to review and discuss the OK and Pitfall solution. The class later is introduced to Nick's and Erica's approaches (the fictional students from the lesson). They discuss Nick's approach briefly. When discussing Erica's solution method, a student, Josef, admits that he had the same approach as Erica.

Notice that the identification made by Josef is with Erica's pitfall solution, not Alicia's. In part, this was because Erica's answer persisted, visually, as something to which he could refer (Alicia erased her original work when she decided she agreed with Pete).

The vignette from Ms. Lee's class can be contrasted with how Ms. Verde used the *MPP* approach in her Grade 7 classroom:

Ms. Verde's Grade 7 Class

Ms. Verde has a student read the problem aloud: "There are 8 times as many giraffes as there are cheetahs. Write an equation to represent the relationship between the numbers of giraffes and the number of cheetahs."

The teacher asks students to think on their own. Once the independent time to think about the problem is over, Ms. Verde tells students to turn over the page and look at the OK solution. She first refers to the work of the fictional student Nick (author of the OK solution for this lesson): "Let's see what Nick did. He is a student like you guys." The class discusses Nick's strategy after the teacher poses the question: "What strategies did Nick use to help make it easier?" First, Myra says, "He did a t-chart." Then Felipe shares, "He did a pattern."

Ms. Verde looks around the room and prompts students to ask Myra questions. She is asked to elaborate and says: "He went 8, 16, 24, and continued the pattern by 8."

In this version of classroom discussion, the class talks about how Nick defined the variables for the problem and if Nick's work makes sense. This allows students multiple entry points into the conversation about why the correct answer is correct. Students who may struggle with one representation have opportunities to articulate their understanding of another representation, scaffolding even reluctant students to participate.

Design Feature 2: Student Thinking in Text and Well-Labeled Diagrams

Research Foundation. Frequently, students in mathematics classrooms present their solutions orally with few visuals. Their explanations may be vague and difficult to follow. This puts visual learners and English language learners who are not instructed in their native language, at a disadvantage (Short, 1991). In the example in Figure 6, fictional student Teresa explains why 2/6 and 1/3 are equivalent, and how she visualized moving shaded parts in the original problem. Key aspects of a concept are presented in print and supported by a drawing.

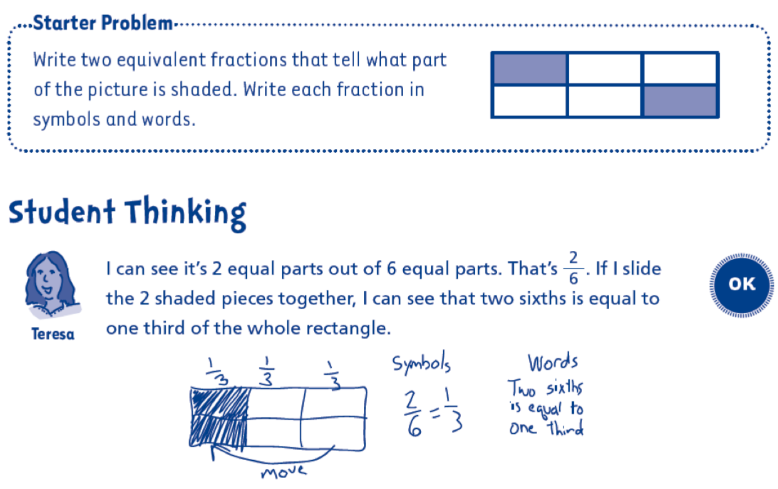


Figure 6. Example Starter Problem visual in OK solution.

Controversy. Do students need all these visuals? What if they just know the answer?

Backstory. This design feature aims at mathematical understanding that includes representing an idea in symbols, words, and pictures. It also disrupts the assumption that visuals are necessary only for students who struggling or are "visual leaners" (Arcavi, 2003; Rapp, 2009; Stylianou, 2002).

Design Feature 3: Include Elements Unlikely to be Surfaced by Students on Their Own

Research Foundation. Sfard (2000) and others point out that sometimes teachers find it difficult to advance learning beyond what students bring to the table. This is particularly common on complex concepts (e.g., comparison involving fractions or proportions as in the OK solution in Figure 7). Beyond extending insights students bring, challenges in understanding student thinking can arise if a teacher's own mathematical understanding is sparse or not well connected, or when many students have pitfalls in their thinking.

Controversy. Why should I focus on an approach or error my students didn't have, instead of one they did?

Backstory. This feature means curriculum designers highlight examples of common student thinking that model particular kinds of reasoning, language, and representations. By "common" we do not mean only that most students will voice it. Rather, these are common in that research and reports of practice have identified a particularly useful (e.g., Figure 7) or especially persistent but problematic way of seeing a mathematical idea or topic that traditional approaches do not fruitfully explore (Lerman, 2001). This design feature includes foregrounding common but often implicit unproductive conceptions. A short continuation of the algebra readiness vignette illustrates how a teacher might toss responsibility for meaning-making back to a student:

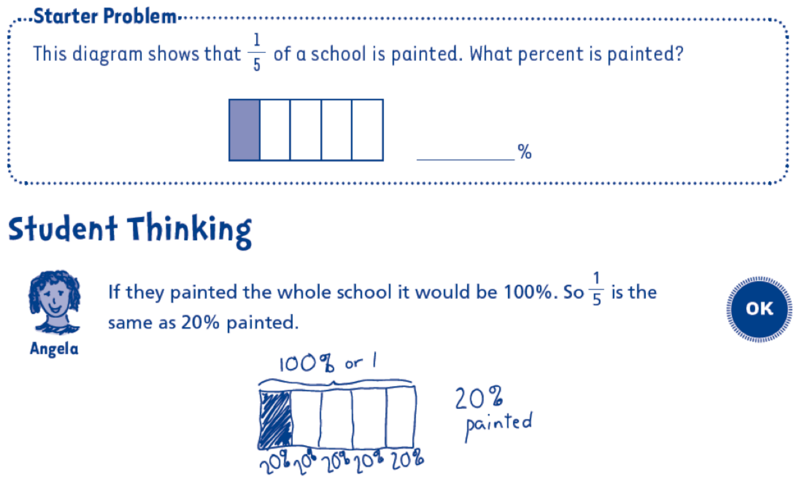


Figure 7. Example Starter Problem Visual in OK Solution.

Ms. Verde's Grade 7 Class, continued

The teacher walks around the room as students work in pairs to make sense of the Pitfall solution. Ms. Verde repeats the statement out loud: "There are 8 times as many giraffes as there are cheetahs..." A student responds, "What does that mean?" The teacher answers with, "That's what I want you to tell me!"

Design Feature 4: Support for Intellectual Risk-taking

Research Foundation. Individual work, which does not need to be shared with the class or even a partner, is the lowest risk structure within an *MPP* lesson (Beghetto & Baxter 2012). After individual think time – and before they are asked to come to the board to explain or justify their ideas – students are meant to work with a partner to talk about particular features of the focal problem or solution. Students are encouraged to engage in higher intellectual risk-taking by coming to the board to explain their ideas and share their own conceptions for the benefit of the class. The *Discussion Builders* poster sentence stems support respectful discussion and a cooperative environment.

Controversy. Teachers are often reluctant to ask students to take intellectual risks in the classroom. Many teachers feel it is important for students to feel comfortable in class *and* think that comfort and safety are the same thing. Though it might cause some discomfort for students if they are "put on the spot" and asked to articulate an idea that is not yet fully formulated, in a classroom environment in which such risk-taking is shared and mutually expected, it can be safe. Also, teachers can be particularly reluctant to ask much of students who are learning English as an additional language or those considered at a lower math level.

Backstory. As Beghetto's (2006, 2016) work has demonstrated, there is a correlation between habits of mind such as risk-taking and higher achievement. In his work, intellectual risk-taking is defined as engaging in adaptive learning behaviors such as sharing tentative ideas, asking questions, and attempting to do or learn new things. Risk-taking in *MPP* classrooms is a gradual process that is encouraged and supported until it becomes an expectation for the shared learning of the class.

> **Mr. Ruiz and his Grade 7 Class**
>
> The teacher begins the class standing next to the *Discussion Builders* poster, "I want you to talk to each other at your tables about which discussion builder you will challenge yourself to use today when we discuss what we cover in class today." Students right away turn and share a selected discussion builder with their partner. The teacher gathers the class together and asks, "Which one are you going to challenge yourself with today?" One student says he has chosen the discussion builder "I'm confused about . . ." and Mr. Ruiz responds, "Oh, so you are going to look for an error or something that is not clear to you. Okay."
>
> After a few other students share their selected discussion builder, the teacher transitions into the math content of the lesson. He introduces the Math Words by appointing students to read a math word aloud to the class. After each sentence is read out loud by a student, Mr. Ruiz encourages a few moments of silence for everyone to make sense of the information that is being read.
>
> Once the students are introduced to the math words in the lesson, Mr. Ruiz gives students 4 minutes of individual time to work on the giraffes and cheetahs Starter Problem (see Figure 5). As students are working independently on the starter problem, the teacher walks around the class encouraging students by using comments such as: "I like that start; nice; I like that work" After a couple of minutes, he quietly says to the whole class, "We can work with whatever types of responses you can think of. It's low risk to try the problem. But no response at all, like I've said before, again it's hard to learn from that, to learn from nothing at all. So, you want to write just a little bit of what you are thinking so you have it to refer to when other people, or you, start sharing ideas.

Notice Mr. Ruiz is explicit about respecting the students' need to first engage in a lower risk interaction with the mathematics, while at the same time encouraging students to stretch and engage in higher intellectual risk-taking. *MPP* professional development includes attention to these features within each lesson so that teachers are clear about the expectations for themselves and their students.

Design Feature 5: Scaffold Discussion for Teachers and Students

Research Foundation. Each Math Pathways & Pitfalls book contains a Discussion Builders poster for students and Clipboard Prompts for teachers (see Figures 3 and 8). The posters realize recommendations from research on classroom discourse development in both first and additional language contexts (Brown, Collins, & Duguid, 1989; Cazden, 2001; Gibbons, 2006). Slightly different for each grade level, the poster contains sentence stems that:

- embed increasingly challenging academic language,
- incorporate support for respectful discussion, and

- promote a proactive stance toward learning — encouraging students to raise questions and make comments without being prompted.

In addition, each *MPP* book contains a video captured in a real classroom with an *MPP* lesson in action. The video is part of the introduction *for students* to the materials. The teacher and students in the video model respectful, productive, and rigorous mathematical conversation. The video gives multiple examples of students choosing and using sentence stems from the *Discussion Builders* poster and of teachers using Clipboard Prompts. The Clipboard Prompts support instructional habits of probing questions and careful listening on the part of the teacher as well as the students (Sawyer, 2004; see Figure 8).

Clipboard Prompts

Effective instructional dialogue digs deep into one student's thinking or one idea at a time. To unfold understanding, begin with a broader question and then follow up with probes for more detail. Use the first set of prompts below to focus students' attention on what the problem means. Use the second set of questions to delve into students' solutions. Finally, use the third set of prompts to invite reflection and ask for other approaches.

1 Understanding the Problem

- ☐ Who can explain what this problem means — not how to solve it — what does it mean?
- ☐ How do you read this (point to a word, phrase, number, symbol, or equation)? What does it mean?
- ☐ Could someone show us how to use a drawing or materials to show what it means? Please explain ______. Please label ______. How does this help us understand?
- ☐ Do you think the answer will be greater or less than ______? Maybe you're not sure yet, but what do you think …?

2 Understanding the Solution Process

- ☐ Who would like to show us how they solved the problem?
- ☐ Please say more. Could you help us understand why you ______? What does ______ mean? Why does it make sense to ______? How is this different from ______?
- ☐ Some of you seem to have questions about this idea. Who has a question? What is confusing?
- ☐ Can someone else help us clarify this idea? Who has another way to help us understand it?

3 Reflecting On and Extending the Problem

- ☐ Explain how you know this answer makes sense. How could you check? How can we prove it is correct?
- ☐ Does someone have another approach? Could we use a drawing? Could we use a paper-and-pencil procedure?

Figure 8. Clipboard Prompts for teachers from *MPP* materials.

Controversy. An explicit "training" process for students that includes modeling and practicing use of *Discussion Builders* and Clipboard Prompts is new to most teachers. Sometimes teachers are hesitant about viewing and discussing the video clip with students or about using the structured collection of conversational scaffolds from the poster and possible in-class prompts.

Backstory. Managing and orchestrating mathematically rich discussions is difficult. Novice facilitators of mathematical discourse often complain that some students do not participate at all. Neither teachers nor their students necessarily understand the components of a discussion, which include speaking, listening, thinking, considering, and agreeing and disagreeing respectfully. Also implicit in much K-8 instruction are the meta-processes involved in forming conjectures and articulating them and in justifying reasoning. Some teachers feel that English learners will be left out of a complex discussion. In fact, after skipping the orientation for students, teachers have reported that they did not anticipate student needs well, it was not easy for most to jump into math discussions. Those same teachers report joy with the successes they find when they take the time to prepare students for expected discourse structures.

Instructional Pathways to Support Language, Discourse, and Equity

It can take teachers some time to overcome their instinct to shift from solving the starter problem to discussing the reasoning behind each example. However, once this is overcome, they are more likely to realize the goal of having many more students participate in ways that require much more student talk than called for in simply producing some solution to it. This is one aspect of *MPP*'s equity strand, to broaden participation in classroom discourse. For students, the aim is no longer whether they get a particular answer. This frees students to explore and discuss pathways to understanding without the stigma of being right or wrong. This is intimately related to another design feature: scaffolding for classroom discussion.

Managing and orchestrating classroom discourse is much more challenging for many teachers than lecturing or modeling a procedure for solving a problem. One of the goals for the *MPP* support materials like the *Discussion Builders* poster and Clipboard Prompts for teachers is to help teachers make the transition to lesson structures that are more demanding and more equitable.

Research suggests that the kind of cognitive apprenticeship and scaffolding included in *MPP* lessons for teachers and students supports new cognitive behaviors and patterns of discourse (see, e.g., Sawyer, 2011). In turn, these structures help English learners gain access to discussions held in English and make the communication more inclusive of all learners. As noted at the beginning, national studies of the impact of the original elementary grades *MPP* materials strongly suggest that English learners benefit as much as first language English speakers from regular participation in *MPP* discussion-rich lessons.

Challenges for Algebra Readiness

In the experimental and quasi-experimental observational studies on use in multiple-subject classrooms of the original *MPP*, teachers were observed to engage in *MPP* practices even when teaching non-*MPP* lessons (both in mathematics and in other subjects; Heller et al., 2010). The design features of *MPP* promoted target student behaviors and interaction that became habitual and central in teacher practice across content areas. However, teachers had the same students for at least a half-day, every day (the original materials were designed for and have seen the most use in multiple-subjects classrooms). Thus, designers faced some challenges in creating *MPP* Algebra Readiness materials when students (and some teachers) move from room to room throughout the day.

In particular, temporal and physical logistics are different in a single-subject setting. Class meetings in middle schools are typically 45 to 50 minutes long and, unlike multiple-subject classrooms, the teacher does not have the option of borrowing a few minutes to make the math lesson longer one day and shorter the next. Work that students or teachers put on a large post-it or on a corner of the board in a multiple-subjects class can persist in the room across several days. However, single-subject classrooms are likely to see different students and different math topics every hour – preserving artifacts of a class discussion is problematic (if possible at all).

Time. As indicated earlier (see Figure 2), in the original *MPP*, lessons are designed to take at least 45 minutes on each of two days. In the version of *MPP-AR* currently being piloted, some of the follow-up in the original design was sacrificed (e.g., Figure 2, final column, does not exist in *MPP-AR*). The design team is considering options for revision – from reshaping timing of lesson components to work across three 35-minute sessions to reconsideration of the role of homework in completing Day 2 *Our Turn* shared problem-solving and *My Turn* independent work.

Physical Space, Math Words, and Artifacts of Discussion. With multiple classes, it can be difficult for single-subject teachers to keep a record of a discussion posted in the room for a week for each of five or more classes. Professional development and in-book suggestions for teachers in *MPP-AR* include ways class-generated materials may be kept by teachers in a variety of formats for re-use. For example, where a class has computers, shared Google Docs or, when there is a classroom document camera, a class folder for items to (re)share. Or, absent technology, a class set of super-sized post-it sheets so that artifacts of the thinking of a particular class can be quickly re-shared on the walls of the room.

In addition to these logistical concerns are the challenges related to instructional habits, classroom cultures, and mathematical knowledge for teaching. Research in these areas has demonstrated there are important similarities and consequential differences in the professional demands of single- and multiple-subject contexts (Ball, Thames, & Phelps, 2008; Marks, 2000). Moreover, more affluent students are more likely to have more knowledgable teachers (Hill, 2007). Given the equity-driven mission of *MPP*, scaffolding for teachers to build their own knowledge continues to be an essential component in *MPP-AR* design.

Culture. In middle school there is often an emphasis on the product (right answer) rather than the journey (discussing the reasoning behind a solution). The *MPP-AR* design response has been rethinking the professional development and in-book teacher materials to explicitly and regularly promote the importance of the journey. Teachers build habits of noticing the pathway and attending to the pitfalls. The key is learning to acknowledge and address failure along the way to building flexibility, fluency, and depth of knowledge for themselves and for students (Smith, 2000).

Teacher Orientation. Even more than elementary school teachers, middle school teachers resist exposing or talking about errors. Also, there is a common perception that visuals are only for students who struggle and more algorithmic ways of solving problems tends to be encouraged (Hiebert & Stigler, 2004). Reluctance to fully engage in using visuals reflects a common teacher concern that the teacher and/or the students find the visuals more confusing than illuminating. However, the ways that *MPP* visuals illuminate ideas is more subtle than a stark spotlight that destroys detail with over-bright simplicity. The purpose of the multiple representations in *MPP* and *MPP-AR* is to support connections among ideas in complex conversations that dig into meaning. The *MPP-AR* live-online professional development tackles this directly. Across several days, teachers review and discuss reports from research and practice as well as view, analyze, and discuss video of *MPP* classrooms and student problem-solving interviews. The explicit goals are to build patience with themselves and students for productive struggle.

Mathematical Knowledge for Teaching. Building discussions requires an ability to think on your feet and relies on teacher understanding of the mathematics itself, of student thinking about the mathematics, and of the language to use to elicit student thinking and explanation. The greatest effort at development of new materials for *MPP-AR* has gone into the content of the student lessons and a new *Mathematical Insights and Teaching Tips* section in the teacher materials for each lesson. These one-half to one-page sections highlight pitfalls from research on student thinking and how to address them in practice. As with the effort to align teacher orientation to the *MPP* approach, these sections aim to increase mathematical knowledge for teaching in several ways: by unpacking and attending separately to *how to do* the mathematics and *how to support students to learn* the mathematics.

As noted in the Grade 7 classroom vignettes included here, when *MPP-AR* instructor materials are explicit about the purpose and intent of each feature in the lessons, teachers can in turn be more explicit and intentional in their practice. Initial observations have shown promise for *MPP-AR* implementation in Grade 7 classrooms. The design team to which the authors belong is field-testing curriculum and teacher professional development materials in the 2018-19 school year and will share further details from that work in the future.

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