Journal of the California Mathematics Project
Volume **8**, Fall 2017

Exploring Area with Lattice Polygons

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Introduction

The concept of area is subtle. When asked for the area of a triangle on a multiple choice test, many students select the perimeter if it is one of the options. Those students rely on memorized formulas without really grasping the concept of area.

Here is an alternative approach to area. It is "naïve" in using hands-on activities that help students to create their own understandings. For simplicity, we ignore questions about areas of curved shapes like circles and of three-dimensional shapes like spheres, concentrating instead on areas of polygons. We will see there is much to be learned from simple explorations like those presented here.

We will rely heavily on two basic properties of area:

> **Property 1.:** If a figure is made up of pieces which do not overlap, its area is the sum of the areas of the parts.
> **Property 2.:** The area of a figure does not change if the figure is moved.

These properties are absolutely fundamental, but they are scarcely mentioned in textbooks for elementary or secondary students (or for teachers)!

In order to measure anything, a unit must be chosen. For area the unit is usually a square of a given size, such as one square centimeter, one square inch, or one square mile. The area of a geometric shape is the number of those square units, possibly broken into pieces and rearranged, that fit into the shape. In view of property 2, if an object consists of several copies of a smaller object, its area is that of the smaller object, multiplied by the number of copies. For example, in Figure 1 the unit square is shown at the upper left. The big rectangle is 4 units long and 3 wide and so contains 12 copies of the unit square. Therefore, its area is 3×4 or 12.

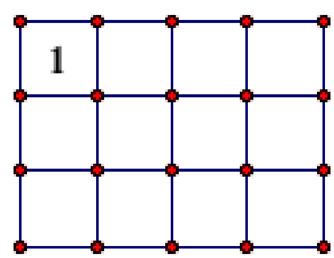

Figure 1

Based on examples like this, we take the area of any rectangle to be the product of its length and width: $A = l \times w$. This reasoning shows why we multiply fractions as we do, because, for example, the product $\frac{3}{4} \times \frac{4}{5}$ is the area of a rectangle $\frac{3}{4}$unit long and $\frac{4}{5}$unit wide.

Figure 2 shows that $\frac{3}{4} \times \frac{4}{5}$ rectangle shaded dark grey. The dark grey rectangle is part of the larger shape, a unit square. The unit square is 1 unit long and 1 unit wide. The unit square in Figure 2 is cut into $4 \times 5 = 20$ identical small rectangles, each of which must have an area of $\dfrac{1}{20}$. The dark grey rectangle contains 3×4 or 12 of the small rectangles, so $\dfrac{3}{4} \times \dfrac{4}{5} = \dfrac{12}{20}$. Similarly, the medium grey, light grey, and white rectangles show, respectively, that:

$$\frac{1}{4} \times \frac{4}{5} = \frac{4}{20}, \qquad \frac{3}{4} \times \frac{1}{5} = \frac{3}{20}, \qquad \text{and} \qquad \frac{1}{4} \times \frac{1}{5} = \frac{1}{20}.$$

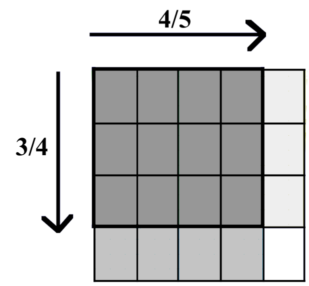

Figure 2

Polygons, a few of which are shown in Figure 3, are figures made of line segments joined end to end. The explorations presented in the remainder of this piece concern polygons with vertices (corners) at points in a square array or lattice (see Figure 4). For this work it is sometimes handy to use rubber bands and geoboards.

Geoboards were used as early as the first half of the 18th century (the blind mathematician Nicholas Saunderson made one). They became popular in schools only when inexpensive, light-weight ones made of plastic became available.

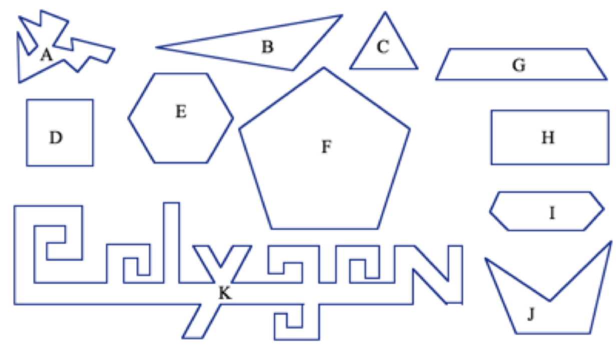

Figure 3

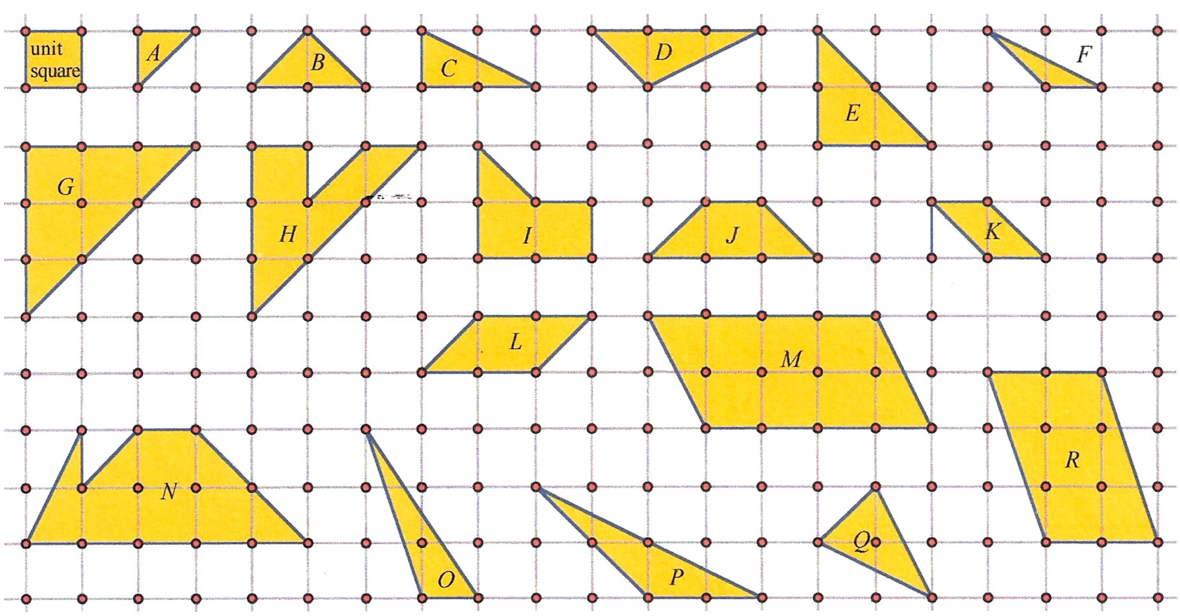

Figure 4

Exploration 1 – A Naïve Introduction to Area

In Figure 4, use the square in the upper left as the unit of area to find the areas of each of the other polygons. Start with A and work through the shapes in Figure 4 in alphabetical order, using earlier answers to help find later ones. Take full advantage of the two basic properties of area mentioned above.

It is interesting to see the varied ways that students can see the same thing. Most students find the area of C by viewing it as half of a 1×2 rectangle, but some find the area by, in effect, rotating the small right-hand part to fill in a complete unit square (see Figure 5).

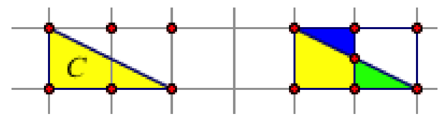

Figure 5

Those same students often extend that thinking to larger shapes, for example, finding the area of triangles E and G by counting unit squares and halves of unit squares in each shape rather than by seeing the triangle as half of a square. They may even find the area of H that way, instead of seeing H as resulting from taking a small half-unit triangle out of G. A triangle like the one in Figure 6 pushes the limits of that kind of thinking; finding the areas of the three small parts does not make it easier to find the area of the triangle. Finding area in this case is easier using "negative space" to see the triangle in Figure 6 as half of a 1×3 rectangle.

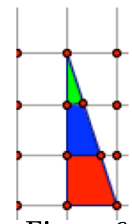

Figure 6

Here are two ways fifth graders handled triangle F. One student first noted that she had already found area A as $\frac{1}{2}$ and area C as 1, as pictured in Figure 7. Next, she thought of C as made up of two shapes, one like F and the other like A, as in Figure 8.

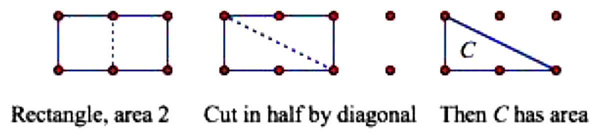

Figure 7

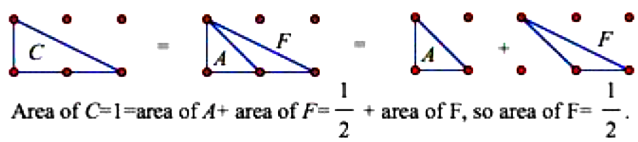

Figure 8

A second student found F hard, so she skipped it and came back to it later. She found a clue in K, which she saw as a rectangle with area 2 minus two triangles like A, one of them upside down, as in Figure 9. She had seen K as made of two copies of F, one of which is upside down. Having found the area of K as 1, she concluded that the area of F must be $\frac{1}{2}$. Other students found still more ways to get the same answer.

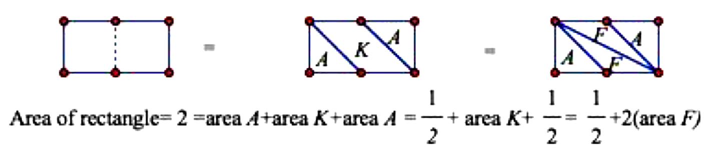

Figure 9

I first used this exercise with a fifth grade class. Some of the students, perhaps through older siblings, had heard that the area of a triangle is half the base times the height. They used that idea to rush to the conclusion that, for example, the area of shape F is $\frac{1}{2}$ without actually thinking through the question. This use of a formula or recipe to get an answer without really thinking through the question is exactly what I intended to prevent! Therefore, we have triangle Q, because neither the base nor the height is easy to find. Students are pushed to figure out the area from first principles. One way to do that is to think of triangle Q as part of a 2×2 square with area 4, as in Figure 10. The four units of area consist of triangle Q, the two flanking larger triangles, and the small triangle in the upper left. That smallest triangle is just like A in Figure 4, so it has area $\frac{1}{2}$, while each of the flanking triangles is a copy of C in Figure 4 and so has area 1. Since the square has 4 units of area, that leaves just 1 and a $\frac{1}{2}$ units of area for triangle Q.

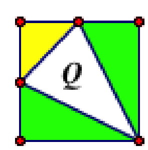

Figure 10

Exploration 2 – Areas of Lattice Squares

In Figure 11, square A is the unit of area. Question: What areas can squares have, if all four of their corners are lattice points? We will call those squares lattice squares.

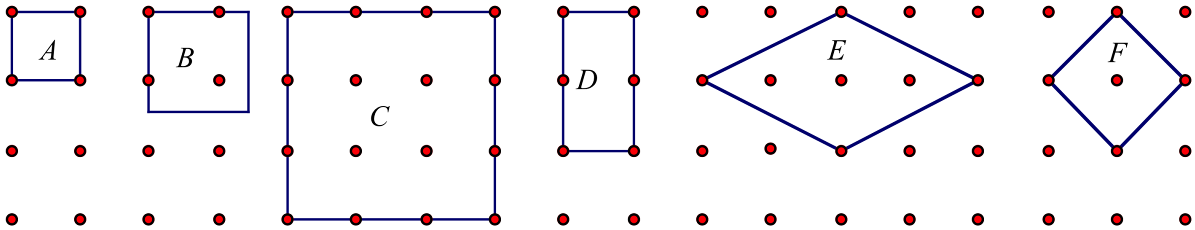

Figure 11

To clarify the question, note that B is not a lattice square, because three of its vertices are not lattice points. D and E are not squares. Students quickly find squares like C and soon guess that the answer to the question is that lattice squares can have perfect square numbers as areas. That only partially answers the question, however. When pressed to investigate further ("Would I have asked you a question with such an obvious answer?" "You need another *slant* on this problem.") they eventually discover examples like F, and then the investigation really gets going.

Now that the question is clear, try to find lattice squares with various areas up to 25. Whether you do this with others or just by yourself, allow plenty of time for this. Graph paper works as well as dot paper.

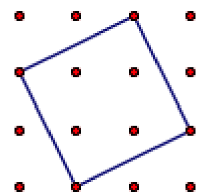

Figure 12

Whenever students claim to have found the area of a lattice square, they have to give a persuasive argument to support that claim. This serves two purposes. First, explaining to others solidifies one's own understanding, and second, they learn the importance of expressing ideas clearly. To illustrate, here are two ways students might justify their claim that 5 is the area of the square in Figure 12.

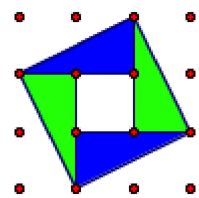

Figure 13

First explanation: Cut the square in Figure 12 into five parts, as in Figure 13. Each shaded triangle in Figure 13 is like triangle C from Exploration 1, together the four shaded triangles together have area 4. The unshaded center square has area 1, so the tilted square has area $4 + 1 = 5$.

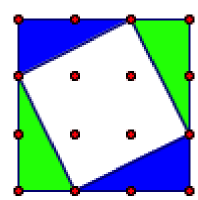

Figure 14

Second explanation: As shown in Figure 14, put a larger square around the given, "tilted," square. The large outer square in Figure 14 is 3 units long and wide, so its area is 9. The four shaded triangles, as in the first method, each have area 1, leaving $9 - 4 = 5$ as the area of the tilted square.

As students find areas they can make, have them tabulate their results, with each whole number from 1 to 25 in the appropriate row, as in Figure 15. We have seen that we can make 5 (shown above) and all perfect squares, so they are already entered in Figure 15. The pattern in the table is hard to find, unless you go back to the geometry.

Can make	**1**			**4**	**5**				**9**							**16**									**25**
Can't make																									

Figure 15

Students may have found that one way to make a lattice square is to put corners at any pair of dots whatever, such as P and Q in Figure 16. Then complete the square based on how far over and up you moved from the first dot to the second.

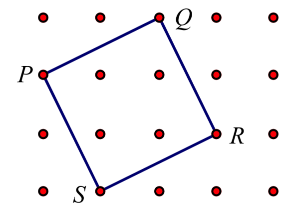

Figure 16

In this case, you get from P to Q by going 2 units to the right and 1 unit up like a knight's move in chess. Now from Q go 2 dots down and one to the right to get to R, and from R go 2 dots to the left and one down to get to S. Finish the square by connecting S to P.

Evidently, the square is determined by the "key move" from P to Q, which we write $(2, 1)$, meaning 2 over, 1 up. A move, such as this is called a **vector**, and the two numbers used to describe a vector are called its **components**. The areas of lattice squares depend on the vectors used to make them. To find the link, start by completing the table below (Figure 17).

Area of lattice square	1	2	3	4	5	6	7	8	9	10	11	12	13
Key move (vector)	(1,0)	(1,1)		(2,0)	(2,1)								

Area of lattice square	14	15	16	17	18	19	20	21	22	23	24	25	26
Key move (vector)													

Figure 17

Now you can see that it is impossible to make a square with area 3, because squares with areas 1 and 2 have used all the key vectors with components 0 or 1, and key vectors with larger components lead to squares with areas of 4 or more. Even at this stage, some students may be baffled. Additional help can come from the observation that, in addition to perfect squares, we can make any number that is one more than a perfect square, such as 2 (1+1), 5 (4+1), 10 (9+1), or 17 (16+1).

Key vectors with a zero component all produce squares with perfect square numbers as areas. The key moves for numbers that are one more than a perfect square have a 1 instead of a 0 as their second component. For example, the key move for the square with area 17 is $(4, 1)$, much like the key vector $(4, 0)$ for the square with area 16, but it has 1 instead of 0 as its second number.

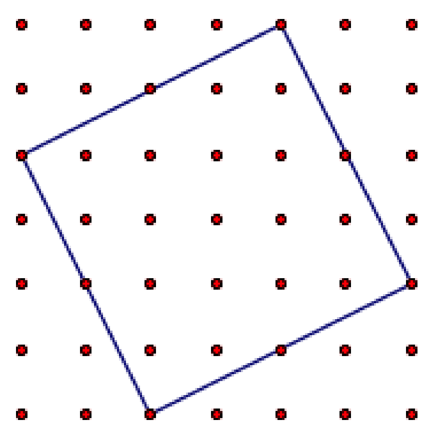

Figure 18

Eventually, students will discover that the area of a square can be found by adding the squares of the components of the key vector for that square. For example, a square with key vector $(4, 2)$, as shown in Figure 18, has area $4^2 + 2^2$ which is $20 + 4$, or 24.

We can check that in two ways – much as was done in Figures 13 and 14.

First check: Divide the interior of the square into pieces as shown in Figure 19. Here the four triangular pieces can be put together in pairs to make two rectangles, each 2×4.

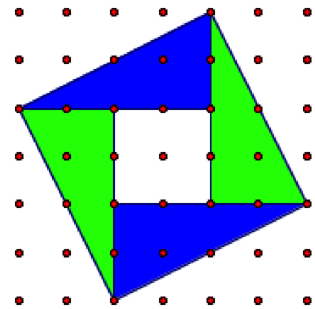

Figure 19

The total area of those two rectangles (four triangles) is 16. The white 2×2 square in the middle has an area of 4. So, the area of the tilted square is $16 + 4 = 20$.

Second check: Embed the tilted square in a larger square, as shown in Figure 20. The large square is 6×6 and so has area 36. As done in the first check, immediately above, the total area of the four triangles is 16. Subtracting the areas of the triangles leaves $36 - 16 = 20$ as the area of the tilted square.

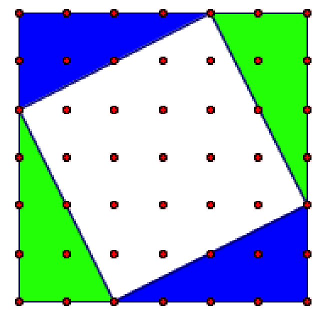

Figure 20

Reflections – What Can We Learn?

Looking back at the work, you may notice that two explanations were used many times and seem to work generally. Consider then, a more general view of the method of embedding the square in a larger one.

Below, in Figure 21, embedded in a larger square whose length and width are $a + b$, is a tilted white square. The smaller, tilted, square is generated by the key vector (b, a). Meanwhile, in Figure 22 we see the same larger square, with the triangles pushed together to form rectangles. Evidently, the two white squares in Figure 22, together, have the same area as the tilted square in Figure 21. So, the tilted square in Figure 21 must have area $a^2 + b^2$. If we call the edge of the tilted square c, its area is c^2 and we have the familiar statement of the Pythagorean relationship, $a^2 + b^2 = c^2$. Note, by the way, that this does not depend on a and b being whole numbers.

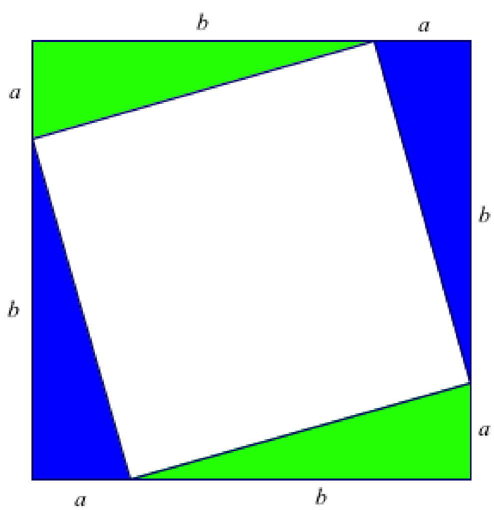

Figure 21

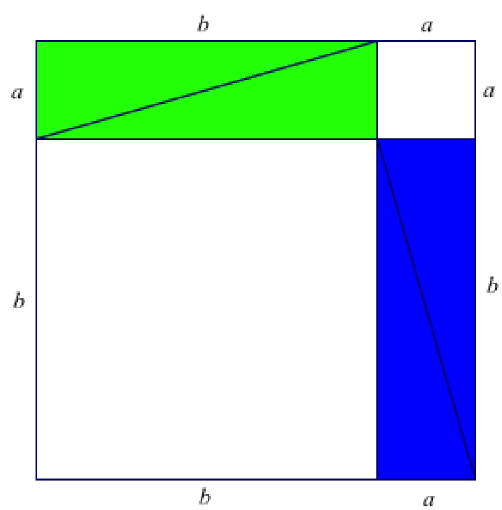

Figure 22

This is, of course, a good way to introduce the Pythagorean theorem, as it avoids having students memorize a formula before they understand it. At a more elementary level, it offers students a peek around the corner at mathematics they will study later in their school careers. We have sidestepped the topic of square roots here, because it is numerical rather than geometric, but this lesson would be a good way to motivate the study of square roots. You might start, for example, with square F in Figure 11. Knowing that it has area 2, can we find the length of each side? That length is clearly between 1 and 2, but determining it precisely is a challenge, because it requires finding a number whose square is 2. That is a wonderful classroom exploration, but we will not pursue that here.

But wait, there's more!

Having, in effect, led students to rediscover the Pythagorean theorem is a worthy outcome of the original investigation, but it does not completely answer the question of what numbers can be the areas of lattice squares. The initial observation that you can make any square whose area is a perfect square number grew to the more complete answer that you can make any square whose area is either a perfect square or a sum of two perfect squares. But just which numbers are those? This is a topic for a whole new exploration, focused on numbers rather than geometry. It turns out that the areas up to 50 that you can make are:

$$1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32, 36, 37, 40, 41, 45, 49, 50.$$

This list of numbers has some curious properties. For one thing, the product of any two numbers in the list is another number in the list. As is sometimes said, this set of numbers is “closed” for multiplication. We have already seen two ways to show that 20 is in the list, but there is a third way, based on the fact that 20 is the product of two numbers in the list, 4 and 5.

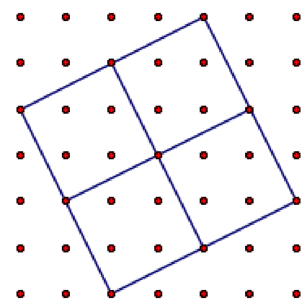

Figure 23

In Figure 23, the big square from Figure 18 is cut into four copies of the square shown earlier to have area 5 (in Figures 13 and 14), so the area of the big square is 4×5 or 20. You might use this idea to show pictorially why 18 is an area of a lattice square (i.e., made up of 9 small tilted squares, each of area 2). That is, $18 = 9 \times 2$, and both 9 and 2 are areas of lattice squares.

The area of the square in Figure 24 can be found by extending the idea used for Figure 23. The larger shaded dots form a coarse tilted lattice. In terms of that coarse lattice, Figure 25 is much like Figure 16 with the area of the larger square equal to 5 basic squares of the coarse lattice. But each basic square of the course lattice has area $2^2 + 3^2$ or 13. So, the area of the square in Figure 24 is 5×13 or 65. As a check, note that in terms of the original, finer, lattice the key move for the large square is $(7, 4)$ and $7^2 + 4^2 = 49 + 16 = 65$.

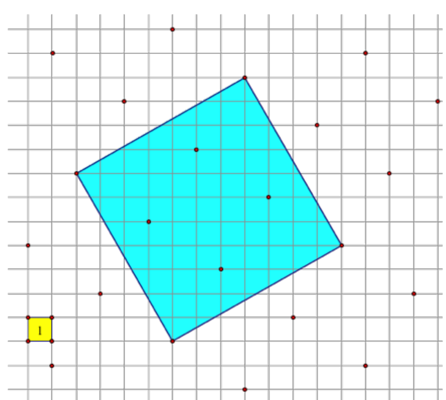

Figure 24

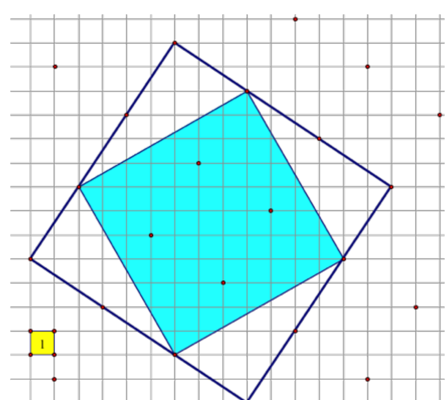

Figure 25

There are other curious properties of the numbers in the list of areas of lattice squares, related to their prime factorizations. If you look at just the odd prime numbers in the list, you will find 5, 13, 17, 29, 37, and 41. Absent from the list are the primes 3, 7, 11, 19, 23, 31, 43, and 47. Of course, the list above goes up only to 50, but that is enough to reveal a pattern. To see it, divide each of these primes by 4 and look at the remainders. Some leave remainder 1, others leave remainder 3... Enough said! By now you can see there is plenty of material for further exploration.

References

Apostol, T. (Producer). *The Theorem of Pythagoras*. This is a 20-minute video with animation by James Blinn and is one of several videos produced by *Project MATHEMATICS!*. It is available in both English and Spanish and can be found on the web, among other places at a site run by NASA.

Beberman, M., Botel, M. Sawyer, W. W. & Wirtz, R. (1967). *Math Workshop for Children.* Chicago, IL: Encyclopedia Britannica Press

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About the Author

Robert Stein <bstein@csusb.edu> taught in New York and Caracas, then at Cal State San Bernardino from 1967–2009, 12 of those years as department chair. He worked with the College Board and the Park City Math Project. Bob served on a local school board from 1989 to 1998 and chaired the Americas Section of the International Study Group on History and Pedagogy of Mathematics for 10 years. Since 2016 he has been the editor-in-chief of the *Journal of the California Mathematics Project* (*JCMP*). Bob and his wife Roni enjoy hiking, playing tennis, travel, cooking, and singing.