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Toward a Conceptual Understanding of Fractions Using The Number Line Model.

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There is a quote from the teaching of Zen to illustrate the journey of seeking deep understanding of concepts.

> *"Before I had studied Zen for 30 years, I saw mountains as mountains, waters as waters. When I arrived at a more intimate knowledge, I came to a place that I saw mountains not as mountains, and waters not as waters. But now I finally obtained the essence of the teaching and I could be at peace, for I saw mountains once again as mountains and rivers once again as rivers"* (Ch'uan Teng Lu).

1. Introduction

When teachers teach a new topic, when students try to learn something and "they cannot wrap their heads around it", they will go through the phase of "seeing mountains not as mountains." Even students comfortable with new concepts are often clumsy and awkward in conveying ideas to others. The tendency to follow, for example, an algorithm by rote memory and without understanding is real and tempting, for confronting misconceptions can be frustrating.

Teachers and students can demonstrate their perseverance by committing to this journey until the new information is fully integrated with prior knowledge, putting their minds once again at peace; when they "see mountains" once again "as mountains."

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2. The Number Line Model; The Unit

Students' fear of fractions is well documented (Ashcraft & Kirk, 2001) and it does not usually show up in the primary grades, when they learn the basic ideas and vocabulary of fractions such as "one-half, two-thirds, and three-quarters." Rather, this fear surfaces when they are taught to add and subtract fractions with different denominators, and the extensions to multiplication and division of fractions.

Fractions research shows that dividing the unit segment into fractional increments extends the number line concept already familiar to students from whole numbers to fractions, enabling learners to visualize the relative positions of fractions, including improper fractions and mixed numbers (Lamon, 2005; Wu, 2008). A learner begins by creating a number line using a paper strip, marking the positions of 0 and 1, to create a unit segment. Next, making a foray into fractions, the learner divides this unit interval, for example, into 10 equal parts, yielding the new unit of measurement of $\frac{1}{10}$. Extending the concept to non-unit fractions, the learner can next visualize two of those units, $\frac{1}{10}$ and another $\frac{1}{10}$, as two $\frac{1}{10}$s or "2$\frac{1}{10}$s" and eventually, a point on the number line called $\frac{2}{10}$. Similarly, the learner can divide the distance between 0 and 1 into 5 equal parts and mark off three $\frac{1}{5}$s or, as we soon see, $3 \times \frac{1}{5} = \frac{3}{5}$. That is, $\frac{1}{5}$ is the new unit of measurement and we have three such lengths on the number line.

Current literature shows that using the number line model facilitates the conceptual understanding of fractions for first time learners in the primary grades, as well as upper grade students, and teachers. According to Lamon, (2005), understanding "unitization" is the key to developing the conceptual understanding of fractions. The principle of unitization enables the learner to connect the prior knowledge of arithmetic of whole numbers to arithmetic of fractions.

How does this knowledge help the learner develop conceptual understanding of the mathematics of fractions? Here is a report on a professional development summer institute program we held last summer, at CSU Stanislaus.

Forty seven Grade 3-5 teachers participated in 40-hours of training using the "Sample Fraction Institute Model" developed by California Common Core State Standards in Mathematics Task Force (CaCCSS-M). One of the findings we learned from this summer institute was that teachers often have a restricted view of fractions, which begets erroneous rules in mathematical reasoning. For example, when we asked the participants in the summer institute "which fraction is closer to 1: 6/7 or 7/6? And why?", ten of 33 participants who gave the right answer also gave us a wrong justification such as "6/7 is closer to 1 because 7/6 is over 1" (answers obtained from the CCMP Pretest, 2012). It is possible that this reasoning stems from prior knowledge that fractions are always less than one, so they eliminated 7/6 as a viable option and chose 6/7 by default. This misunderstanding was observed repeatedly in the various class settings of pre-service teacher classes and in-service professional development. During the first day of the "Fraction Institute" when we asked the participants to provide an example of a fraction, the majority of responses were of the numbers less than 1.

Failing to include improper fractions as examples of fractions can lead to a narrow and restricted view of this important topic. To help the learners expand their views on the different types of fractions, we propose to use number line model to teach fractions.

3. Sample Lesson 1: Locate Fractions on the Number Line

The first lesson of learning fractions using the number line model is to help learners see a fraction, say, $\frac{1}{2}$, $\frac{3}{4}$, or $\frac{4}{3}$ as a point on the number line. At our summer Fraction Institute, participants first created a number line with whole numbers by first marking the unit segment on the number line and then using this unit of measurement to locate the numbers 2, 3 and 4. Next, they constructed another identical number line placed it parallel to the first and divided the line segment between 0 and 1 into 3 equal parts and used the new unit of $\frac{1}{3}$ to mark the rest of the number line. As participants marked the fractions $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{3}$, etc. on the number lines, they noticed that 1 and $\frac{3}{3}$ were the same point on the line, as were 2 and $\frac{6}{3}$, 3 and $\frac{9}{3}$. (See Figure 1.)

The use of the number lines helped students to expand their understanding of fractions to mixed number and improper fractions. Furthermore, use of the number line model transformed the concept of equivalent fractions from an abstract algorithm to concrete pictorial representations. Using the number line models, learners can rename fractions using the principle of unitization and then applying the schema of whole numbers to locate proper and improper fractions as well as the mixed numbers (See Figure 1). The following sample lesson shows how the participants applied the principle of unitization to locate fractions.

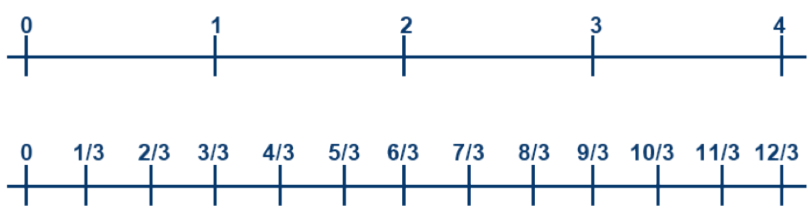

Figure 1: Locate the fractions on the number line.

Figure 2 shows a set of examples that challenged learners to investigate their present understanding of fractions. The task was to locate the fractions A, B, and C on the lines. At first, the majority of participants assumed that A, B, and C had the same value because they were on the same vertical line (same point) of three parallel and seemingly identical number lines. Upon further introspection, however, participants noticed the relative position of 0s and 1s on each number line. They noticed that two of fractions A and C were less than 1, and the fraction B was greater than 1. Thus, the number line model helped participants understand and visualize the value of a mixed number or improper fraction as a fraction whose value is greater than 1.

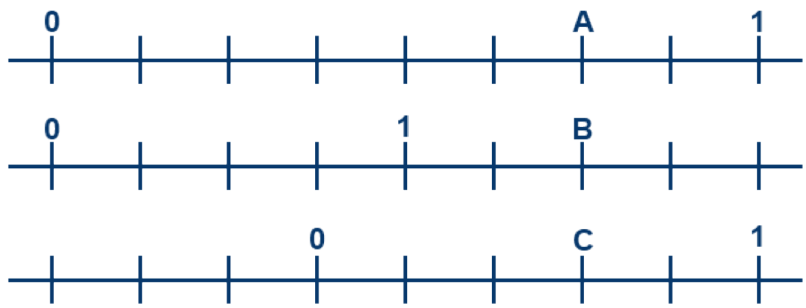

Figure 2: Find the fractions A, B, and C on the Number Lines

When asked how to determine the fractions of A, B, and C; here is a paraphrasing of what some said: To locate A, B, and C on the number lines, one must apply the knowledge of denominators and numerators. This helps to determine the "unit of measurement" in each number line. This is an application of unitization. For example, in the first line, there are eight equal parts between 0 and 1 where each part has a value of $\frac{1}{8}$. The point A is at the sixth mark after 0; therefore, it is at the position of $6 \times \frac{1}{8} = \frac{6}{8}$. In the second line, there are four equal parts between 0 and 1 where each part is $\frac{1}{4}$. The point B is at the sixth mark after 0, therefore, is at the position $6 \times \frac{1}{4} = \frac{6}{4} = \frac{3}{2} = 1\frac{1}{2}$. Although A and B appear to have the same numerator, they are of different values because the unlike denominators yielded in the different units of measurement in the respective number lines. Thus, to answer this question correctly, learners must have understanding on how to determine the unit of measurement on the number line.

The traditional pizza or pie analogy can be proportionally incorrect because the teachers were not able to easily partition the fractions in the proper units. They did not have a solid understanding of unitization and had a hard time judging whether or not answers they got were reasonable.

To measure anything requires a unit. Consider the non-example of putting a pile of dust on top of a second pile of dust. What do we get? Not two piles of dust, but a large pile of dust. This is because the concept of a pile is not a well-defined unit of measurement, unlike " one inch," "one apple," or "one a minute." The juxtaposition of examples and non-examples of "unit" gives the learner a better understanding of the principle. The learner will understand that the principles of unitization apply in the domain of whole numbers *and fractions*, but it does not apply in the domain of piles of dust, or other undefined units.

4. Sample Lesson 1: Locate Fractions on the Number Line

To add or subtract fractions, the learner must first transform the fractions into the same unit of measurement, after which the arithmetic of whole numbers can be applied. Just as we can rename one hour as 60 minutes, using unitization, the learner can rename a fraction. For example, 1 can be renamed as $\frac{3}{3}$ or $\frac{5}{5}$ or $\frac{100}{100}$. One-half can be renamed as $\frac{2}{4}$, $\frac{3}{6}$, $\frac{5}{10}$, or $\frac{10}{20}$. If learners understand the concept of unit and unitization, they will have conceptual understanding of

(1) the why and how of adding and subtracting fractions with like denominators,
(2) the why and how of converting fractions with unlike denominators to ones with like denominators, and
(3) the ability to articulate the rationale.

5. Sample Lesson 2: How to Add or Subtract Fractions with the Number Line Model: Find $\frac{4}{3} - \frac{1}{2}$ using the number line.

Figure 3a illustrates the same unit on the number line with three different sub-units: $\frac{1}{3}$, $\frac{1}{2}$, and $\frac{1}{6}$. With these common sub-units, the answer to the subtraction is easy: $\frac{4}{3} - \frac{1}{2} = \frac{8}{6} - \frac{3}{6} = \frac{5}{6}$
To check our answer we use the inverse of subtraction to go over the process on the number line in the reverse order. See Figure 3b without the hash marks. $\frac{4}{3} - \frac{1}{2} =?$ is the same as $\frac{4}{3} = \frac{1}{2} + ?$. The answer is displayed on the figure: $\frac{1}{2} + \frac{5}{6} = \frac{3}{6} + \frac{5}{6} = \frac{8}{6} = \frac{4}{3}$.

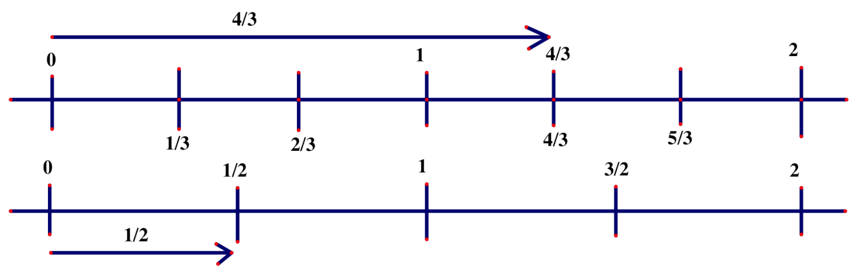

Figure 3a: Setting up the solution of the subtraction problem.

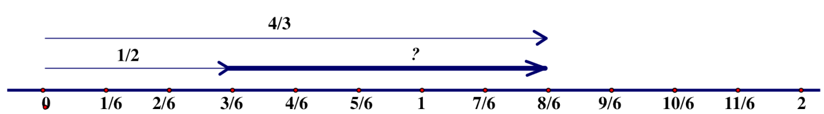

Figure 3b: Solution of the subtraction problem.

6. Sample Lesson 3: Multiplication and Division of Fractions

In addition to the adding or subtracting fractions with unlike denominators, the learner is also prone to feel challenged by multiplication and division of fractions conceptually. Multiplication of fractions, such as $\frac{2}{3} \times \frac{3}{4}$, can be represented with an area model and might be easier for the learner to understand. (See (Wu, 2008) for more details.)

Divisions of fractions, lacking a relevant schema to facilitate understanding seems the most foreign operation to teachers and students. In the absence of conceptual understanding, many teachers will use mnemonic of "KFC" ("Keep" the first one, "Flip" the second one, and then "Change" the division sign to multiplication) to teach students the procedure of solving division of fractions.

Unfortunately, without the conceptual understanding of the division of fractions this strategy could be over-simplified and confusing because they do not remember which fraction they should "flip." Furthermore there is not mathematical operation called "flip." More importantly, most teachers have difficulty explaining to the students *why* they should flip the second fraction in solving the division of fractions. This was part of the feedback from participants of our summer institute.

To overcome this learning barrier, the California Task Force suggests teachers use the number line model to help students make sense of the division of fraction problems.

6.1. Sample Lesson 3: How to Teach Division of Fractions: Find $2 \div \frac{2}{3}$ and $\frac{5}{8} \div \frac{1}{4}$ using the number line.

(1) Locate 2 and 2/3 on the number line. (See Figures 4a, 4b)
(2) Apply the arithmetic of whole numbers to the arithmetic of fractions: How many times will $\frac{2}{3}$ fit into 2?
(3) Use the inverse of division, which is multiplication or repeated addition, to check if the answer is correct.

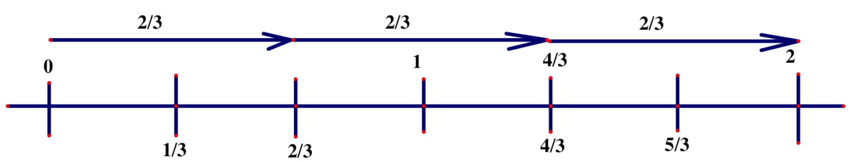

Figure 4a: How many $\frac{2}{3}$ can fit into 2?

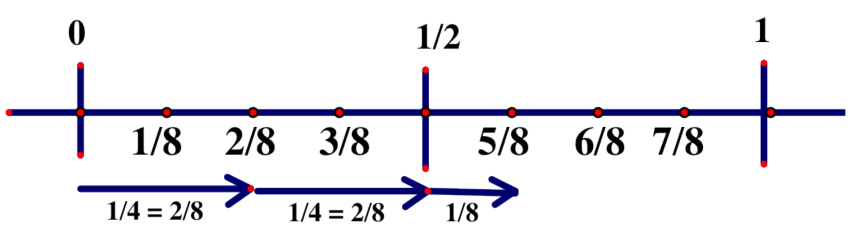

Figure 4b: How many $\frac{1}{4}$ can fit in $\frac{5}{8}$?

In both examples above, teachers can turn division of fraction, from an abstract expression, into a concrete example by using the number line.

Method: use two paper strips as the manipulatives. The learner can find how many $\frac{2}{3}$ of 1 unit can fit into 2, 1 units. The learner can also demonstrate the understanding that "the second number" in both problems is the new unit of measurement. To engage in academic discourse, the learner must know its proper name: *divisor.*

Moreover, the tangible notion of a fraction comparing the parts present to the parts that make a whole is typically a problem when students are told that a fraction is merely a division problem with the dividend called the numerator and the divisor called the denominator. Educators should compare the two notions, showing their common result. For example, the division notion of $15 \div 3$ is the number of groups of 3 that comprise 15. The fraction notion of $\frac{15}{3}$ is 15 units of $\frac{1}{3}$. Since 3 units of $\frac{1}{3}$ comprise 1, as the number line conveys, the task reduces to the division notion with the result of 5.

The ability to name the numbers correctly will help students in the process of conveying what they know or ask clarifying questions as needed. The learner can also use their prior knowledge and proficiency in the first language (division of whole numbers) to help them understanding a new concept and their second language (fractions).

7. Connection to the Common Core Standards for Mathematical Practice

For many learners, the mathematics of fractions is a foreign concept. The terminology and syntax used in the math discourse at times appear to be a second language. The understanding of unitization helps the learners to bridge the understanding of arithmetic of whole numbers (the schema of their first language) to arithmetic of fractions (similar schema presented in their second language). The uses of the number line help learners to think in pictures. When the learners have the right pictures in their heads, and words to accurately describe pictures in their heads, they gain proficiency in thinking, speaking, and reasoning with the mathematics of fractions.

In this process, the learner demonstrates the ability to "model with mathematics" (Common Core Standards for Mathematical Practice, Standard 4), "attend to precision" (Standard 6), "look for and make use of structure" (Standard 7), and "look for and express regularity in repeated reasoning" (Standard 8).

When a teacher engages students in mathematics practice as outlined in the common core standards, the teacher is guiding students to develop an in-depth conceptual understanding so as to "make sense of problems" (Standard 1) and "construct viable arguments and critique the reasoning of others" (Standard 3). When learners communicate with others about the process of finding equivalent fractions, they demonstrate their perseverance in problem solving and their ability to use language to convey their conceptual understanding.

Using the number line model, teachers can teach the specific abstract concepts including unitization, adding and subtracting fractions, multiplication or divisions of fractions with concrete manipulative (e.g. uses of paper strips) or pictorial representation. Helping students move flexibly between concrete examples, pictorial representations, symbols and mathematic formula of fractions can help students deepen their understanding and their ability to engage in academic discourse about fraction with others.

In the practice of discourse, teachers and students both deepen their understanding of fractions and values and uses of the number line models. In other words, they develop conceptual fluency of fraction sense, developing critical thinking skills while paving the road to success in algebra and geometry. At the end, the learners not only once again see the mountains as mountains; they can also describe the wonders they experienced while journeying through the mountains.

To effectively teach the concepts and uses of fractions using the number lines, teachers must integrate the four levels of depth of knowledge about fractions as defined by (Webb, 2005). In other words, teachers must first provide students a wide range of examples of fractions and show the minimal difference between fractions and non-fractions. Second, teachers should adopt the two-prong approach: (i) helping students develop the conceptual understanding of fractions by connecting it to their prior knowledge of whole numbers and real-life connections, and (ii) helping students to develop procedural fluency and conceptual fluency in making sense of fractions problems using the number line models.

To help the students with diverse learning needs, including students with limited English proficiency and math vocabulary, teachers need to model how to use the vocabulary correctly in the academic discourse. Teachers need to use the "think aloud" (van Someren, Barnard, & Sandberg, 1994) to make their cognitive and reasoning process overt to the learner. Ultimately, teachers need to understand that teaching fractions using the number lines is one of the vehicles to teach students to the Common Core Standards of Mathematical Practice.

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