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Modeling a Swinging Atwood Machine

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Abstract. The motion of a Swinging Atwood Machine is difficult to solve for using Newtonian Mechanics. Lagrangian Mechanics, on the other hand, is an extremely useful tool for solving this system. In this lab we find the Lagrangian for a Swinging Atwood system, solve for the equations of motion, and compare our model to that of the observed motion of the system. Our model provides a good approximation of the motion, with small discrepancies due to the unknown mass of our pulley and dissipative forces.

The Atwood Machine was invented by Rev. George Atwood as a simple way to demonstrate classical mechanics. It is composed of two masses connected by a string placed over a massless, frictionless pulley. The pulley will rotate such that the combined gravitational potential energy of the masses decreases over time. With the only forces on the masses being gravity and the constraint force from the string, the acceleration of the masses can be used to determine the value of g if the ratio of the masses is known.

We look at an Atwood Machine with slightly more complex initial conditions in this lab. One of the two masses is not confined to one dimensional motion, but is allowed to freely swing. This apparatus is unsurprisingly called the Swinging Atwood Machine. Fig. 1 shows a diagram of the Swinging Atwood Machine. In this lab we use a video of the Swinging Atwood Machine, and we attempt to predict and model its motion.

The motion of the mass in the Swinging Atwood Machine can be difficult to solve for using Newtonian Mechanics. This is due largely to the constraint force of the string, which varies with time. Luckily, the problem is easily solved using Lagrangian mechanics for a system in polar coordinates (r, ϕ). Lagrangian Mechanics has two crucial advantages to Newtonian Mechanics. First, Lagrangian Mechanics enables us to ignore the constraint forces in solving

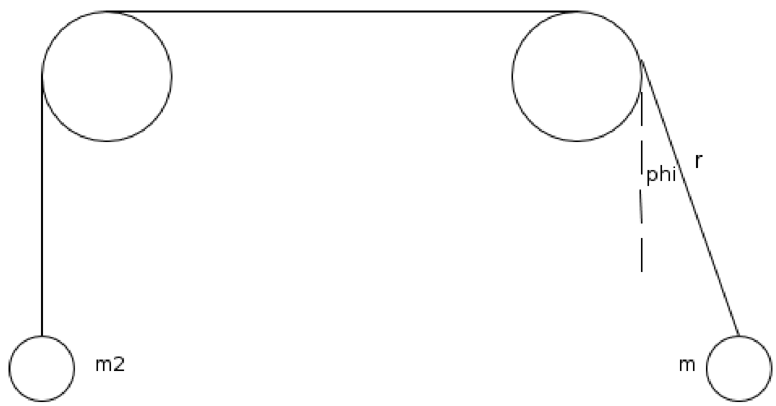

Figure 1: A string connects the two masses in our Swinging Atwood Machine. This string is layed over two pulleys, and the system is allowed to move freely under the force of gravity. One bob of mass m_2 is free to move vertically, while the other bob of mass m is allowed to move radially and tangentially from the point where the string touches the pulley. Phi (ϕ) is defined as the angle between the string and the vertical, while r is the distance between m and the pulley. We assume massless and frictionless pulleys, as well as a massless and non-extendable string.

for the equations of motion. Second, Lagrangian mechanics allows us to choose generalized coordinates, greatly simplifying the problem.

The first step to solving for the equations of motion is to determine the Lagrangian of the system, defined as

$$\mathcal{L} = \mathrm{T} - \mathrm{U}, \tag{1}$$

where T is the kinetic energy of the system and U is the potential energy[1]. The potential energy of the system, solely due to the gravitational potential energy for each of the masses, is

$$\mathrm{U} = m_2 gr - mgrcos(\phi). \tag{2}$$

The kinetic energy of the system is the sum of the kinetic energies of each mass. Both kinetic energies are found by using

$$\mathrm{T} = \frac{1}{2} m\dot{r}^2. \tag{3}$$

For the non-swinging mass, the kinetic energy is found by applying Eq. 2 without any dependence on ϕ. The kinetic energy of the swinging mass is dependent on both the velocity in both the radial and tangential directions. The total kinetic energy for the system is thus,

$$\mathrm{T} = \frac{1}{2}m_2\dot{r}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2), \tag{4}$$

making the Lagrangian

$$\mathcal{L} = \frac{1}{2}m_2\dot{r}^2 + \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - m_2gr + mgrcos(\phi). \tag{5}$$

With the Lagrangian in hand, we can use the Euler-Lagrange equations for each coordinate to solve for the motion of the object. The Euler-Lagrange's equations for this system are

$$\frac{\partial\mathcal{L}}{\partial r} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) = 0, \tag{6}$$

and

$$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) = 0. \tag{7}$$

Solving for Eq. 6 gives us

$$mr\dot{\phi}^2 - m_2g + mgcos(\phi) = (m + m_2)\ddot{r}. \tag{8}$$

Knowing that acceleration in the radial direction in polar coordinates is $\ddot{r} - r\dot{\phi}^2$, we can rearrange Eq. 8 to give us the radial force

$$m(\ddot{r} - r\dot{\phi}^2) = F_r = -m_2g + mgcos(\phi) - m_2\ddot{r}. \tag{9}$$

We can also solve Eq. 7 in the same way to give us

$$m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = F_{\phi} = -mgsin(\phi). \qquad (10)$$

Thus, now we have the force equations and can model our system.

Data and Results. Modeling the motion of our Swinging Atwood is now possible with the force functions known. Our first step is to place a video of a Swinging Atwood Machine into a tracking and modeling program. The program we use for this experiment is Tracker[2]. Tracker has the ability to take a video of an object in motion and graphically track the object's motion. The video we input into Tracker contains only the swinging portion of the Swinging Atwood[3], with a piece of polar graph paper behind it for convenience when it comes to calibration. Once properly calibrated, Tracker will measure both the observed radius r and angle ϕ over time.

Tracker also has the ability to create a model of an object's motion and predict where the object will be at a particular time. By inputting the equations of motion given by Eq. 9 and Eq. 10, Tracker will show the predicted postition of the mass (r, ϕ). The only things that need to be added here are the initial conditions, measured to be $\phi_0 = 25$ degrees, $\omega_0 = 0$ deg/s, $r_0 = 0.48$ m and $\dot{r_0} = 0$ m/s.

One major piece of information missing from our model at this point is the ratio of m to m_2. This makes inputting Eq. 9 and 10 problematic as we don't know the values of two of the constants. This problem is easily resolved by modeling the data with different values for the masses until the model comes close to fitting. Our model creates a close approximation to the motion in the video with $m_2/m = 1.055$. Fig. 2 shows the results of our comparison between the observed motion and the model for the radius of the Swinging Atwood, while Fig. 3 shows the results for the angle.

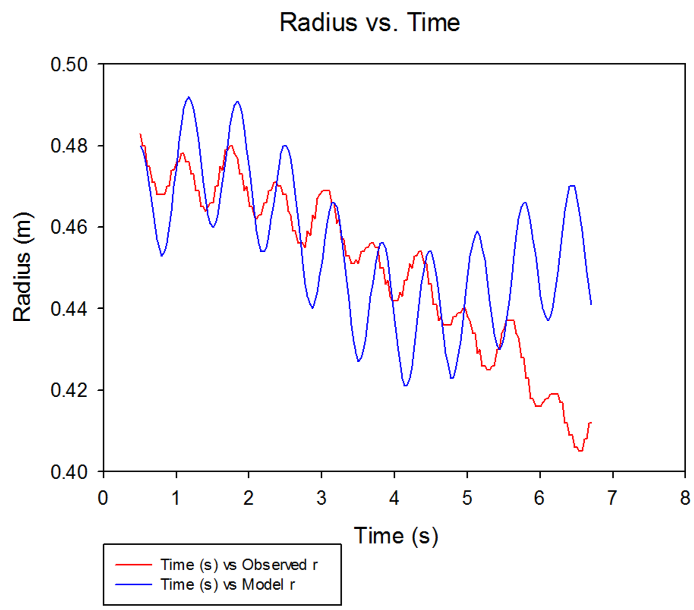

Figure 2: Our model for the radius of the Swinging Atwood has some of the same qualitative features as what is observed, yet there are clear discrepancies. The radius oscillates for both the observed data and the model, but the amplitude of the oscillation is larger in the model. Both also see their average radius decrease for the first part of the motion, while only the model has its average radius begin to rise again.

Our model, though close, does not perfectly predict the motion of the mass. Both our observed data and model in Fig. 2 show that there is an oscillation in the radial position of mass m, but the oscillation has a larger amplitude in the model. We can also see the average radius decreased for the first part of the motion in both masses, while only the model has its average radius increase again. In Fig. 3, we see that the angular positions of mass m are very similar between the observed data and the model. However, we can see a small decrease in the amplitude of the observed data with respect to the model, as well as a slightly higher frequency of oscillation for the model.

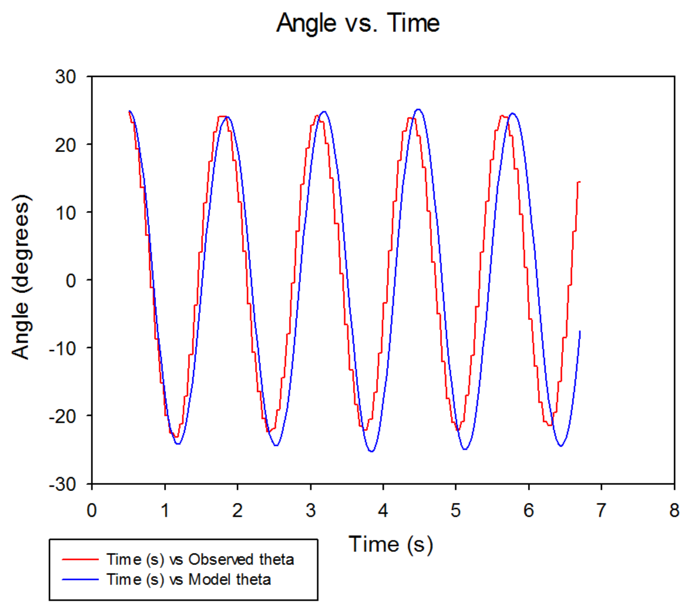

Figure 3: Our model for the angle of the Swinging Atwood is quite accurate. The amplitude and phase of the oscillations remain roughly the same throughout the time of measurement. The only noticable differences are a small decrease in the amplitude of the observed motion with respect to the model and a slightly higher frequency of oscillation for the model.

Our model was built on the assumption of a massless and frictionless pulley, as well as a massless and un-extendable string, neither of which holds in reality. The diminished amplitude of oscillation in the radius for our physical object in Fig. 2 can be attributed to the non-zero force it takes to accelerate the mass of the pulley, and our model's lack of dissipative forces. The different frequency and diminishing amplitude of the observed motion in Fig. 3 are akin to that of a damped oscillator. Thus, we find that perhaps our model requires a damping term. The lack of a return to a longer average radius in the physical system in Fig. 2 can be attributed to a dissipated energy in the system.

Despite the small discrepancies, our model is a good approximation for the motion of a Swinging Atwood Machine. We were able to model the motion of the swinging mass in the Swinging Atwood Machine simply by determining the kinetic and potential energy of the system. This is no easy accomplishment when one only has Newtonian Mechanics in their toolbox. When compared with the task of fully modeling this system in Newtonian Mechanics, we have shown Lagrangian Mechanics to be a sufficient and convenient tool to model mechanical systems, so long as future models of the Swinging Atwood Machine should include dissipative forces.

References: [1] Taylor, John R. (2005). Classical Mechanics. University Science Books.

[2] Doug Brown, Tracker: http://www.cabrillo.edu/ dbrown/tracker/.

[3] Leah Ruckle, Swinging Atwood's Machine Model: http://www.compadre.org/ops/items/detail.cfm?I